Chapter 2

Chromatic-Accidental Arithmetic

2.1 Pitch Labeling: Naming the Notes

We start from the very beginning. In music, we often hear letters being thrown around. When it comes to notes, there are seemingly only 7 letters.

Consider the musical alphabet:

$$\mathcal{L} = \{A, B, C, D, E, F, G\}$$

Definition 2.1.1 (Musical Alphabet)**.** *The musical alphabet is defined as a set of letters $\mathcal{L} = \{A, B, \ldots, F, G\}$, where the letter after G loops back to be A again, and so on.*

Note ($\mathcal{L} \cong \mathbb{Z}_7$)**.** *$\mathcal{L}$ is isomorphic (equivalent in structure) to the group of remainders after dividing any integer from $\mathbb{Z}$ by 7, the set which we call the field $\mathbb{Z}_7 = \{0, 1, \ldots, 5, 6\}$. In other words, it is a relabeling of $\mathbb{Z}_7$. In this group, we can more clearly see the previously stated fact about* ***looping*** *if we express we fix $A = 0, B = 1, \ldots, G = 6$ and observe $6+1 = 7 \cong 0$ corresponds to G being followed by A.*

2.1.1 Letter Degrees & Sequences

In turn, we are ready to define letter-scale degrees.

Definition 2.1.2 (k-Degree Letter)**.** *The k^{th} degree letter from a given letter $L \in \mathcal{L}$ is denoted the value $\Lambda_k(L)$, which is defined as $L \oplus (k-1)$. In other words, it is the letter obtained from taking the final element of the sequence $\mathbb{L}_k[L]$.*

Definition 2.1.3 (k-Degree Letter Sequence)**.** *The k^{th} degree letter sequence from a given letter $L \in \mathcal{L}$ is denoted the set $\mathbb{L}_k[L]$,*

$$\mathbb{L}_k[L] = \{L, L \oplus 1, \ldots, L \oplus (k-1)\}$$

Or, if we let $\Lambda_j(L) = L \oplus j$, the equivalent sequence:

$$\mathbb{L}_k[L] = \{\Lambda_j(L) \mid j \in \mathbb{N}\}_{j=0}^{k-1}$$

where $\oplus$ represents taking $(k-1)$ more steps in the musical alphabet, ***with looping****. The emphasis is in this small part here, which gives us non-trivial relations. As such, we obtain k letters in total if we include our original starting letter (L), so we call it the k^{th} scale degree.*

Example (5^{th} from F)**.** *The fifth-degree letter from F is the terminus of the sequence:*

$$\mathbb{L}_5[F] = \{F, G, A, B, C\}$$

Or in other words, $\Lambda_5(F) = C$.

Example (4^{th} from C)**.** *The fourth-degree letter from C is the terminus of the sequence:*

$$\mathbb{L}_4[C] = \{C, D, E, F\}$$

Or in other words, $\Lambda_4(C) = F$.

Example (3^{rd} from C)**.** *The third-degree letter from C is the terminus of the sequence:*

$$\mathbb{L}_3[C] = \{C, D, E\}$$

Or in other words, $\Lambda_3(C) = E$.

Example (6^{th} from E)**.** *The sixth-degree letter from E is the terminus of the sequence:*

$$\mathbb{L}_6[E] = \{E, F, G, A, B, C\}$$

Or in other words, $\Lambda_6(E) = C$.

Example (7^{th} from A)**.** *The seventh-degree letter from A is the terminus of the sequence:*

$$\mathbb{L}_7[A] = \{A, B, C, D, E, F, G\}$$

Or in other words, $\Lambda_7(A) = G$.

Example (2^{nd} from G)**.** *The second-degree letter from G is the terminus of the sequence:*

$$\mathbb{L}_2[G] = \{G, A\}$$

Or in other words, $\Lambda_2(G) = A$.

Note that the patterns that are emerging might be surprising. They were in fact chosen intentionally, to show the following theorem.

Theorem 2.1.1 (Rule of 9)**.** *The rule of nine tells us the following:*

$$\Lambda_k(L) = L' \iff \Lambda_{9-k}(L') = L$$

To put it more simply, we pair fifths & fourths, sixths & thirds, and sevenths & seconds. This could be attributed to the octave-partitioning scale axiom, since the union of these letter templates gives us diatonic scales.

Example (F-C-F)**.** *The **fifth**-degree letter of F is the letter C. The **fourth**-degree letter of C is F. Indeed, $5 + 4 = 9$. Together, their union forms this particular diatonic letter scale partition:*

$$\{F, G, A, B, C\} \cup \{C, D, E, F\}$$

Example (C-E-C)**.** *The **third**-degree letter of C is the letter E. The **sixth**-degree letter of E is C. Indeed, $3 + 6 = 9$. Together, their union forms this particular diatonic letter scale partition:*

$$\{C, D, E\} \cup \{E, F, G, A, B, C\}$$

Example (A-G-A)**.** *The **seventh**-degree letter of A is the letter G. The **second**-degree letter of G is A. Indeed, $2 + 7 = 9$. Together, their union forms this particular diatonic letter scale partition:*

$$\{A, B, C, D, E, F, G\} \cup \{G, A\}$$

L	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	$L_{(1)}$
A	B	C	D	E	F	G	A
B	C	D	E	F	G	A	B
C	D	E	F	G	A	B	C
D	E	F	G	A	B	C	D
E	F	G	A	B	C	D	E
F	G	A	B	C	D	E	F
G	A	B	C	D	E	F	G

(**Scales, Revisited**)

Recall that we defined a scale as a set of intervals that lie between a frequency and one octave above it. This takes the form of a (Do, ..., Do) set of sonorities in Solfege. This definition of a scale is quite handy, and is in fact intertwined with the reasoning behind choosing 7 letters in $\mathcal{L}$. In fact, it is by construction that we do since we want each letter once to occur in a scale until we reach the same letter again (representing the same pitch an octave above). We want our scales to end at the same note, an octave above. As such, this naturally leads to needing 8 notes in total to go back to the same letter.

Note (Etymology of Octave)**.** *Alas, we have obtained the etymology of the word octave! That is, the word octave has the root word* $8-$ *(oct-). This is connected since an 8-letter note sequence is a transposition of the musical alphabet – or in other words, a diatonic scale partitioning an octave!*

With this, we define a generic heptatonic scale template, which is merely a transposition of $\mathcal{L}$ starting on any letter.

Definition 2.1.4 (Heptatonic Scale Template)**.** *Suppose we have* $L \in \mathcal{L}$*. The heptatonic scale template is the letter sequence:*

$$\mathcal{S}_7[L] = \mathbb{L}_7[L] := \mathcal{L}_{(L)}$$

We could also denote $\mathbb{L}_7[L] = \mathcal{L}_{(L)}$*, which is the musical alphabet, beginning at* L*. As such, the scale goes up until it reaches the same pitch an octave above, where the next iteration of the sequence repeats (also an octave above).*

Example (F Heptatonic Scale)**.** *The F heptatonic scale template is given by:*

$$\mathcal{S}_7[F] = (F, G, A, B, C, D, E)$$

To notate a scale repeating an octave above, we use an index in parenthesis to indicate the number of octaves transposed above. So, for instance, the scale would repeat:

$$\mathcal{S}_7[F] = (F, G, A, B, C, D, E, F_{(1)}, G_{(1)}, \dots)$$

Note that this gives us the octave relationship between our notes. For any $L \in \mathcal{L}$,

$$\frac{\mathrm{Freq}(L_{(k)})}{\mathrm{Freq}(L)} = 2^k = \square_k$$

2.1.2 Accidentals

Definition 2.1.5 (Accidentals)**.** *An accidental $\alpha \in \mathcal{A}$, where $\mathcal{A} = \{\flat, \natural, \sharp\}$ is a symbol that introduces a notion of distance, or arithmetic to the space of the musical alphabet $\mathcal{L}$. Namely, it does so in accordance to the semitonal distance function d_1, defined below.*

(**Pitch Labels**)

Finally, we are able to formally construct notes as we know them. Recall that we defined two important objects coming from two sets:

(1) Letters, from the **Musical Alphabet**;

$$L \in \mathcal{L} = \{A, B, \ldots, F, G\}$$

(2) Accidentals, from the **Single-Accidental Alphabet**;

$$\alpha \in \mathcal{A} = \{\flat, \natural, \sharp\}$$

Combining these two constructs together naturally motivates the definition of a note-letter. Let us denote the space of pairs of letters and accidentals $\mathcal{L}^* = \mathcal{L} \times \mathcal{A}$

Definition 2.1.6 (Note-Letter)**.** *A pitch label, note-letter, or more formally, an accidentalized alphabet letter, is defined by pairing an alphabet letter $L \in \mathcal{L}$ and an accidental $\alpha \in \mathcal{A}$. Then, an accidentalized letter-note pair is simply the pairing π:*

$$\pi = (L, \alpha) = L^\alpha \text{ where } (L, \alpha) \in \mathcal{L}^*$$

Note (Accidental Neutralization)**.** *Note that flats and sharps neutralize each other. In other words, any note-letter L satisfies $(L^\flat)^\sharp = L^\natural = (L^\sharp)^\flat$*

2.2 White-Key Note-Letter Distances

Before we begin, we formalize a notion mentioned above in our discussion of heptatonic scales.

Distance Rule 1 (Octave Distance)**.** *The octave distance is defined for a letter and its octave-transposed copies, valued in $\mathbb{Z}$. In other words, it tells us the distance between a* ***fixed*** *alphabet letter $L \in \mathcal{L}$, and a $k \in \mathbb{Z}$ octave-transposed version of that same letter, $L_{(k)}$. Namely, we have:*

$$d_{\square}(L, L_{(k)}) = 12k$$

In other words, it's the linear function $f(k) = 12k$, where $k \in \mathbb{Z}$ is the number of octaves transposed. It basically just tells us, go down or up by 12 or -12 semitones for each octave. This rule is not super important, but we define it for completeness.

Distance Rule 2 (Semitonal-Accidental Norm)**.** *The semitonal distance is defined as a binary operation $d_\alpha : \mathcal{L} \times \mathcal{L}^* \to \{-1, 0, 1\}$. In other words, it tells us the distance between a* ***fixed*** *alphabet letter $L \in \mathcal{L}$, and an accidentalized version of that same letter, $L^\alpha \in \mathcal{L}^*$. Fixing $\alpha \in \mathcal{A}$:*

$$d_\alpha(L, L^\alpha) = |\alpha|$$

where we have:

$$|\alpha| = \begin{cases} -1, & \alpha = \flat \\ 0, & \alpha = \natural \\ +1, & \alpha = \sharp \end{cases}$$

Or in other words:

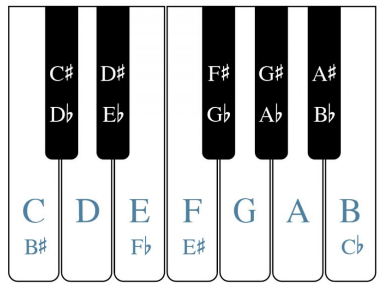
$$|\flat| = -1, |\natural| = 0, |\sharp| = +1$$

Example (C, $C^\sharp$)**.** *The distance between the notes C and $C^\sharp$ is $|\sharp| = +1$. So, $d_\alpha(C, C^\sharp) = 1$.*

Example (D, $D^\flat$)**.** *The distance between the notes D and $D^\flat$ is $|\flat| = -1$. So, $d_\alpha(D, D^\flat) = -1$.*

2.2.1 Accidentals & Color

Consider the diagram below:



Definition 2.2.1 (Black Flat-Supported Letter)**.** *A letter $L \in \mathcal{L}$ is called a black flat-supported letter if* $Color(L^\flat) = \boldsymbol{Blk}$*. That is,*

$$L \in \overline{\boldsymbol{Blk}}[\flat] \iff Color(L^\flat) = \boldsymbol{Blk}$$

The flat-supported keys are denoted $\overline{\boldsymbol{Blk}}[\flat]$ *and are:*

$$\overline{\boldsymbol{Blk}}[\flat] = \{A, B, D, E, G\}$$

And their complement is the white flats, $\overline{\boldsymbol{Wht}}[\flat]$*:*

$$\overline{\boldsymbol{Wht}}[\flat] = \mathcal{L} \setminus \overline{\boldsymbol{Blk}}[\flat] = \{C, F\}$$

Note (Flat Pentatonics)**.** *Note that* $\overline{\boldsymbol{Blk}}[\flat]$ *could be more easily thought of as the notes that spell out* $G^\flat$ *pentatonic major or* $E^\flat$ *pentatonic minor.*

Definition 2.2.2 (Black Sharp-Supported Letter)**.** *A letter $L \in \mathcal{L}$ is called a black sharp-supported letter if* $Color(L^\sharp) = \boldsymbol{Blk}$*.*

$$L \in \overline{\boldsymbol{Blk}}[\sharp] \iff Color(L^\sharp) = \boldsymbol{Blk}$$

The sharp-supported keys are denoted $\overline{\boldsymbol{Blk}}[\sharp]$ *and are:*

$$\overline{\boldsymbol{Blk}}[\sharp] = \{A, C, D, F, G\}$$

And their complement is the white sharps, $\overline{\boldsymbol{Wht}}[\sharp]$*:*

$$\overline{\boldsymbol{Wht}}[\sharp] = \mathcal{L} \setminus \overline{\boldsymbol{Blk}}[\sharp] = \{B, E\}$$

Note (Sharp Pentatonics)**.** *Note that* $\overline{\textbf{\textit{Blk}}}[\sharp]$ *could be more easily thought of as the notes that spell out* $F^\sharp$ *pentatonic major or* $D^\sharp$ *pentatonic minor.*

Definition 2.2.3 (Den Letter)**.** *A letter* $L \in \overline{\textbf{\textit{Den}}}$ *is a den key if*

$$Color(L^\flat) = Color(L^\sharp) = \textbf{\textit{Blk}}$$

So, L is a den key when $L \in \overline{\textbf{\textit{Blk}}}[\flat]$ *and* $L \in \overline{\textbf{\textit{Blk}}}[\sharp]$*, meaning it is both black-sharp and black-flat supported. Or in other words,* $L \in \overline{\textbf{\textit{Blk}}}[\flat] \cap \overline{\textbf{\textit{Blk}}}[\sharp]$*, the intersection of the two sets. The den keys are denoted* $\overline{\textbf{\textit{Den}}}$ *and are:*

$$\overline{\textbf{\textit{Blk}}}_D = \{A, D, G\}$$

We call them den keys since the natural key versions look like dens in a piano, surrounded by two black keys.

We memorize the den keys with the mnemonic:

$$\overline{\textbf{Blk}}_D = \{A, D, G\} = \text{A Den ``G''}$$

perhaps alluding to the fact that $G \in \overline{\textbf{Blk}}_D$.

Next, we define the complement of the den keys to be the **white-neighbouring keys**

Definition 2.2.4 (White Neighbours)**.** *A letter* $L \in \mathcal{L}$ *is a white neighbour if*

$$Color(L^\flat) \textbf{ \textit{or} } Color(L^\sharp) = \textbf{\textit{Wht}}$$

So, L is a white neighbour ($L \in \overline{\textbf{\textit{WhtNeb}}}$) *when* $L \notin (\overline{\textbf{\textit{Blk}}}[\flat] \cap \overline{\textbf{\textit{Blk}}}[\sharp])$*,*

$$L \notin (\overline{\textbf{\textit{Blk}}}[\flat] \cap \overline{\textbf{\textit{Blk}}}[\sharp]) \iff L \notin (\overline{\textbf{\textit{Blk}}}[\flat]) \textbf{ \textit{or} } L \notin (\overline{\textbf{\textit{Blk}}}[\sharp])$$

meaning it isn't both black-sharp and black-flat supported. The white neighbours are denoted $\overline{\textbf{\textit{WhtNeb}}} = \mathcal{L} \setminus \overline{\textbf{\textit{Den}}}$ *and are:*

$$\overline{\textbf{\textit{WhtNeb}}} = \{B, C, E, F\}$$

We call them white neighbours since each one directly borders another white key.

With all of this in place, we are finally able to compute distances between consecutive letters! Or in other words, start doing arithmetic exclusively on the **white notes** first.

2.2.2 Note Steps

Distance Rule 3 (Step Distance)**.** *The step distance is defined as another rule for the distance function,* $d_2 : \mathcal{L} \rightarrow \{1, 2\}$*. Consider a fixed letter* $L \in \mathcal{L}$*, and its right neighbour, or its second-degree letter* $\Lambda_2 = L \oplus 1$*. The distance between two consecutive, unaccidentalized alphabet letters can now be computed using the above notions of black-accidental support.*

$$d_2(L) = d(L, \Lambda_2) = k \ \ \textit{where}\ 1 \leq k \leq 2$$

where we have:

$$d_2(L) = d(L, \Lambda_2) = \begin{cases} 1, & \Lambda_2 \in \overline{\boldsymbol{Wht}}[\flat] \\ 2, & \Lambda_2 \in \overline{\boldsymbol{Blk}}[\flat] \end{cases}$$

Equivalently,

$$d_2(L) = d(L, \Lambda_2) = \begin{cases} 1, & L \in \overline{\boldsymbol{Wht}}[\sharp] \\ 2, & L \in \overline{\boldsymbol{Blk}}_\sharp \end{cases}$$

So, notice the relationship:

$$L \in \overline{\mathbf{Wht}}[\sharp] \iff \Lambda_2 \in \overline{\mathbf{Wht}}[\flat]$$

$$L \in \overline{\mathbf{Wht}}[\flat] \iff \Lambda_2 \in \overline{\mathbf{Wht}}[\sharp]$$

Example $(A,\ B)$**.** *Since* $A \in \overline{\boldsymbol{Blk}}[\sharp]$ *or* $B \in \overline{\boldsymbol{Blk}}[\flat]$*, the distance* $\delta_2(A) = \delta(A, B) = 2$*.*

Example $(E,\ F)$**.** *Since* $F \in \overline{\boldsymbol{Wht}}[\flat]$*, the distance* $\delta_2(E) = \delta(E, F) = 1$*.*

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	B	2	w
B	C	1	h
C	D	2	w
D	E	2	w
E	F	1	h
F	G	2	w
G	A	2	w

Sorting them out,

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	B	2	w
C	D	2	w
D	E	2	w
F	G	2	w
G	A	2	w
B	C	1	h
E	F	1	h

Now, we are able to define enharmonics.

Definition 2.2.5 (Chromatic Enharmonic)**.** *The two spellings of a note in between two letters a whole tone apart, accidentalized in the appropriate directions, are equivalent and called proper enharmonics. They are:*

$$\{(A^\sharp, B^\flat), (C^\sharp, D^\flat), (D^\sharp, E^\flat), (F^\sharp, G^\flat), (G^\sharp, A^\flat)\}$$

The white-key (improper) enharmonics are:

$$\{(B, C^\flat), (E, F^\flat), (B^\sharp, C), (E^\sharp, F)\}$$

Den Keys. Notice the role of den keys here!

In general, we can now define a general distance function δ for any degree letter.

Distance Rule 4 (n-Degree Distance)**.** *The n^{th}-degree distance is defined as another rule for the binary operation $\delta : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}_{12}$. Consider a fixed, unaccidentalized letter $L \in \mathcal{L}$, and its n^{th}-degree letter $\Lambda_n = L \oplus (n-1)$. We obtain its distance by iteratively finding the distances between the two steps along the way. Letting $L = \Lambda_0$,*

$$\delta_n(L) = \delta(L, \Lambda_n) = \sum_{j=0}^{n-1} d_2(\Lambda_j) = \sum_{j=0}^{n-1} d(\Lambda_j, \Lambda_{j+1})$$

2.3 Tertiary Harmony

2.3.1 Thirds/Skips and Dyads

(**Dyads and Triads**)

Finally, we are ready to define quality dyads - one of the most useful objects you will ever encounter. As we know, the words **major** and **minor** are ubiquitous in music theory. We will refer to them as tonal "qualifiers" where they add a quality to some tonal center. What is a note without its third?

Definition 2.3.1 (Quality)**.** *Quality is a binary indicator that tells us whether a chord is major or minor. This is only unambiguous for dyads and triads. We denote a quality* $\mathcal{Q} \in \tilde{\mathcal{Q}}$ *where* $\tilde{\mathcal{Q}} = \{\Delta, \nabla\}$*.*

By composing two steps, we obtain a skip.

Distance Rule 5 (Skip Distance)**.** *The skip distance is defined as another rule for the distance function,* $d_3 : \mathcal{L} \to \{3, 4\}$*. Consider a fixed, unaccidentalized letter* $L \in \mathcal{L}$*, and its third degree letter* $\Lambda_3 = L \oplus 2$*. We obtain its distance by iteratively finding the distances between the two steps along the way.*

$$d_3(L) = d(L, \Lambda_3) = k \;\; \textit{where } 3 \leq k \leq 4$$

where we have:

$$d_3(L) = d(L, \Lambda_3) = d_2(L, \Lambda_2) + d_2(\Lambda_2, \Lambda_3)$$

Recall that d_2 *has a range of* $\{1, 2\}$*, since no two consecutive letters lie in* $\overline{\boldsymbol{Wht}}[\flat]$ *(we cannot find 3 consecutive white keys in a piano, only a maximum of 2),* d_3 *has a range of* $\{3, 4\}$*. Recall* $\overline{\boldsymbol{Wht}}[\flat] = \{C, F\}$ *so regardless of what we choose as* L*, the smallest degree we can get where both letters are in* $\overline{\boldsymbol{Wht}}[\flat]$ *is* 4*.*

Next up, classifying dyads:

$$\mathcal{Q}_3(L) = \mathcal{Q}(L, \Lambda_3) = \begin{cases} \nabla, & d(L, \Lambda_3) = 3 \\ \Delta, & d(L, \Lambda_3) = 4 \end{cases}$$

Well, since we are working with a small number of steps, there are few enough possibilities that we could conceivably count them. If we want to classify the ways to obtain skips from steps, we can only use the following arithmetic partitions of the numbers 3 and 4 (allowing only 1 and 2)

$$3 = 1 + 2 = 2 + 1$$

$$4 = 2 + 2$$

Corresponding to the interval patterns:

$$t = hw = wh$$

$$T = ww = w^2$$

L	Λ_3	$d_3(L, \Lambda_3)$	Symbol
A	C	3	∇
B	D	3	∇
C	E	4	Δ
D	F	3	∇
E	G	3	∇
F	A	4	Δ
G	B	4	Δ

Sorting them out,

L	Λ_3	$d_3(L, \Lambda_3)$	Symbol
A	C	3	∇
B	D	3	∇
D	F	3	∇
E	G	3	∇
C	E	4	Δ
F	A	4	Δ
G	B	4	Δ

Definition 2.3.2 (Quality Dyad)**.** *Consider a letter $L \in \mathcal{L}$. A quality dyad is the pair of a note-letter on L and its third $T = \Lambda_3(L)$. In other words:*

$$\pi = (L^\alpha, T^{\alpha_T})$$

In other words, it is an assignment of an incidental pair (α, α_T) *onto a pair of notes, a root and its third. This induces the next rule for semitonal distance. Namely, if the quality of the quality dyad is major, then:*

$$d(L^{\alpha}, T^{\alpha_T}) = 4 \implies \kappa_2 \Delta[L^{\alpha}] = T^{\alpha_T}$$

Example (A, $C^{\sharp}$ Major Dyad)**.** *The A major dyad is given by the note pair* $\pi = \kappa_2 \Delta[A] = \Delta_2[A] = (A, C^{\sharp})$*.*

Definition 2.3.3 (Complementary Dyad)**.** *Consider a letter* $L \in \mathcal{L}$*. If we have a quality dyad* $\mathcal{Q}_2(L)$*, then the complementary dyad is the dyad* $\mathcal{Q}^c(\Lambda_3)$*. If* $Q = \Delta$*, its complementary quality is minor, which we denote* $Q^c = \nabla$*. Or vice versa, if* $Q = \nabla$*, then* $Q^c = \Delta$*. Recall that:*

$$|\mathcal{Q}^c| = 7 - |\mathcal{Q}|$$

Rule: A Triad is the sum of any dyad and its complement.

$$\nabla[A] \oplus \Delta[C] = \int_A^{\varphi} C$$

Rule: A Quality Seventh Chord is a stack of a dyad and its complementary chain.

$$\nabla^7[A] = \nabla[A] \oplus \Delta[C] \oplus \nabla[E] = \int_A^{\varphi} C \oplus \int_C^{\varphi} E$$

Alternatively, using triad-bass pairs:

$$\nabla^7[A] = \nabla[A] \oplus \Delta_3[C] = A \setminus \Delta_3[C]$$

Definition 2.3.4 (Quality Triad)**.** *Consider a letter* $L \in \mathcal{L}$*. A quality triad is the triple of note-letters on* L *including its third* $T = \Lambda_3(L)$ *and fifth* $F = \Lambda_5(L)$*. In other words:*

$$\pi = (L^{\alpha}, T^{\alpha_T}, F^{\alpha_F})$$

In other words, it is an assignment of an incidental triple $(\alpha, \alpha_T, \alpha_F)$ *onto a triple of notes: a root, its third, and its fifth. However, this time the letter* F^{α_F} *is predefined to be the complementary quality dyad to the root-third dyad, creating a perfect fifth. In other words, the major triad is given by:*

$$\Delta_3 = (L, T = \Delta_2(L), F = \nabla_2(T))$$

And the minor triad:

$$\nabla_3 = (L, T = \nabla_2(L), F = \Delta_2(T))$$

Alluding to the fact that:

$$3 + 4 = 4 + 3 = 7$$

In summary:

$$Q_3 = (L, Q_2(L), Q_2^c(T))$$

2.3.2 White-Key Interval Complementation

(**Steps and their Parity Complements**)

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	B	2	w
B	C	1	h
C	D	2	w
D	E	2	w
E	F	1	h
F	G	2	w
G	A	2	w

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	$B^\flat$	1	h
C	$D^\flat$	1	h
D	$E^\flat$	1	h
F	$G^\flat$	1	h
G	$A^\flat$	1	h
B	$C^\sharp$	2	w
E	$F^\sharp$	2	w

(**Thirds and their Quality Complements**)

We do the same for thirds:

L	Λ_3	$d_3(L, \Lambda_3)$	Symbol
A	C	3	∇
B	D	3	∇
C	E	4	Δ
D	F	3	∇
E	G	3	∇
F	A	4	Δ
G	B	4	Δ

L	Λ_3	$d_3(L, \Lambda_3)$	Symbol
A	$C^\sharp$	4	Δ
B	$D^\sharp$	4	Δ
D	$F^\sharp$	4	Δ
E	$G^\sharp$	4	Δ
C	$E^\flat$	3	∇
F	$A^\flat$	3	∇
G	$B^\flat$	3	∇

2.3.3 Two Skips make a Fifth/Triads

By composing two thirds/skips, we obtain a fifth or a triad.

Distance Rule 6 (Triadic Distance)**.** *The triadic or fifth distance is defined as another rule for the binary operation $d_5 : \mathcal{L} \times \mathcal{L} \to \{6, 7\}$. Consider a fixed, unaccidentalized letter $L \in \mathcal{L}$, and its fifth degree letter $\Lambda_5 = L \oplus 4 = \Lambda_2 \oplus 2$. We obtain its distance by iteratively finding the distances between the two skips/thirds along the way. Note, we only have diminished and perfect fifths between any two white keys. Augmented chords are not naturally occurring, in this sense.*

$$d_5(L) = d(L, \Lambda_5) = k \ \ \text{where } 6 \leq k \leq 7$$

where we have:

$$d(L, \Lambda_5) = \begin{cases} 6 = |\{\nabla, \nabla\}| \\ 7 = |\{\nabla, \Delta\}| = |\{\Delta, \nabla\}| \end{cases}$$

In general, we use the equality:

$$d_5(L) = d(L, \Lambda_5) = d_3(L, \Lambda_3) + d_3(\Lambda_3, \Lambda_5)$$

L	Λ_5	$d_5(L, \Lambda_5)$	Symbol
A	E	7	φ
B	F	6	φ°
C	G	7	φ
D	A	7	φ
E	B	7	φ
F	C	7	φ
G	D	7	φ

Sorting them out,

L	Λ_5	$d_5(L, \Lambda_5)$	Symbol
A	E	7	φ
C	G	7	φ
D	A	7	φ
E	B	7	φ
F	C	7	φ
G	D	7	φ
B	F	6	φ°

And the corresponding complementary fixes/adjustments:

L	Λ_5	$d_5(L, \Lambda_5)$	Symbol
A	$E^\flat$	6	φ°
C	$G^\flat$	6	φ°
D	$A^\flat$	6	φ°
E	$B^\flat$	6	φ°
F	$C^\flat$	6	φ°
G	$D^\flat$	6	φ°
B	$F^\sharp$	7	φ

2.4 Fifths and Furry Cats

2.4.1 The Story of the Furry Cats

Remember the story of the Furry Cats:

Furry **C**ats **G**et **D**ancey **A**round **E**very **B**ird

Now, what do Furry Cats have to do with music theory? Well, this strange but cute mnemonic gives us an insight into how our white keys are tuned! Consider the following recursive chain of letter fifths:

$$F - C - G - D - A - E - B$$

In other words, we have the sequence:

$$\mathcal{F} = \{\lambda_0 = F, \lambda_1 = C, \ldots, \lambda_6 = B\}$$

Using our letter cycle definition:

$$\mathcal{F} = \mathcal{C}_{7,5} = \{\lambda_{j+1} = \Lambda_5(\lambda_j) : \lambda_0 = F\}_{j=0}^{5}$$

Using index notation for compositions of letter degrees:

$$\mathcal{F} = \{\Lambda_5^{(c)}(F)\}_{c=0}^{6}$$

where $\Lambda_k^{(2)}(L) = \Lambda_k(\Lambda_k(L))$.

This chain is special, which we will call the furry-cat chain. It is special because it is the **unique** and **largest possible** chain of fifths that we can generate before needing **accidentals**. In other words, letting $\varphi = q_7 \in \mathbb{I}_{\mathcal{E}}[12]$ be our perfect fifth generator in our semitone space $\mathbb{Z}_{12}$, we obtain the canonical white key stack of fifths, which we call F Lydian.

Theorem 2.4.1 (Furry Cat Theorem)**.** *The Furry Cat Theorem states that the largest non-accidentalized possible chain of fifths is the sequence of white keys denoted by the Furry Cat Sequence. In other words, the Furry Cat sequence $\mathcal{F} = \{\lambda_i = (L_i, \natural) \mid L_0 = F\}$ where $d_{\mathcal{T}}(L_i, L_{i-1}) = q_7 = \varphi$ or more simply, has a **consecutive semitone distance** of a Perfect Fifth:*

$$d(L_i, L_{i-1}) = 7$$

Note (White-Note & Natural Accidentals)**.** *It is overdue to comment that any letter of the musical alphabet with the natural accidental $\alpha = \natural$ is considered a white note in the piano template.*

2.4.2 Circle of Fifths

Recall that $d_5(B, F) = 6$. By the semitone distance function, we know we could fix this by taking $F \mapsto F^\sharp$ so that we may have $d_5(B, F^\sharp) = d_5(B, F) + d_1(F, F^\sharp) = 6 + 1 = 7 = \varphi$

However, now, we have $d_5(F^\sharp, C) = d_5(F, C) - d_1(F, F^\sharp) = 7 - 1 = 6$. So, to readjust, we do $C \mapsto C^\sharp$. This chain reaction motivates the tile of fifths.

Finally, we obtain the god-forsaken circle of fifths. By using the accidental bridge principle, we have the chain:

$$\mathcal{F}^\flat - \mathcal{F}^\natural - \mathcal{F}^\sharp$$

And it yields the tiled circle of fifths:

$$F^\flat - C^\flat - G^\flat - D^\flat - A^\flat - E^\flat - B^\flat$$

$$F^\natural - C^\natural - G^\natural - D^\natural - A^\natural - E^\natural - B^\natural$$

$$F^\sharp - C^\sharp - G^\sharp - D^\sharp - A^\sharp - E^\sharp - B^\sharp$$

Note that we only have twelve unique notes, so there are enharmonic spellings.

In summary:

$$\mathcal{F} = \mathbb{S}_5^{(6)}(F) = \Phi_6[F]$$

The circle of fifths shall therefore be called:

$$\mathcal{F}^* = (\mathcal{F}^\flat \cup \mathcal{F} \cup \mathcal{F}^\sharp) = \Phi_{12}$$

The circle of fourths:

$$\mathcal{R}^* = (\mathcal{R}^\sharp \cup \mathcal{R} \cup \mathcal{R}^\flat)$$

2.5 Chromatic Pitch Neighbours

2.5.1 Accidental Transposition

This whole time, we were discussing arithmetic on $\mathcal{L}$. How do we extend our arithmetic to $\mathcal{L}^*$? The answer? Mapping note letters from $\mathcal{L} \times \mathcal{L}$ that are distance-equivalent in $\mathcal{L}^* \times \mathcal{L}^*$.

Theorem 2.5.1 (Accidental-Transpositional Preservation Principle)**.** *Suppose we have two unaccidentalized letters $L, L_* \in \mathcal{L}$ which are a distance $\tilde{d}(L, L_*) = k$ semitones apart. Then, if we fix an accidental $\alpha \in \mathcal{A}$, accidentalizing both letters by the same accidental α preserves the distance. In other words,*

$$\tilde{d}(L, L_*) = k \iff \tilde{d}(L^\alpha, L_*^\alpha) = k$$

Note that here we use the function $\tilde{d}$, which automatically assumes the correct form depending on what degree $j \in \{1, 2, \ldots, 7\}$ we have for $L_* = \Lambda_j(L)$. In a way, this theorem is a form of a **sufficiency** result on the white keys or unaccidentalized letter space.

Definition 2.5.1 (k-Degree Letter Stack)**.** *The k^{th} degree letter stack sequence on a root $L \in \mathcal{L}$ is denoted $\mathbb{S}_k^{(N)}[L]$ and is given by recursively taking N iterations of Λ_k starting at L. So, letting $\Lambda^{(0)} = L$:*

$$\mathbb{S}_k[L] = \{\Lambda_k^{(j+1)} = \Lambda_k(\Lambda_k^{(j)})\}_{j \in \mathbb{N}}$$

Example (Thirds from F)**.** *We have:*

$$\mathbb{S}_3[F] = \{F, A, C, E, G, B, D, F, A, C, \ldots\}$$

2.5.2 Pitch Distances in Any Key

Definition 2.5.2 (Chromatic Left-Neighbour)**.** *The chromatic left-neighbour chord is the chord obtained by shifting every pitch in it down by one semitone, or effectively **flattening** it. So, given a chord κ with pitches:*

$$\kappa = \{\pi_1, \pi_2, \ldots, \pi_N\}_{N \in \mathbb{N}}$$

We compactly write this:

$$\kappa \odot \flat = \{\pi_i \odot \flat\}_{i=1}^{N}$$

More simply,

$$\kappa \odot \flat = \{(\pi_1 \odot \flat), (\pi_2 \odot \flat), \ldots, (\pi_N \odot \flat)\}$$

We similarly define right-neighbours, as one might expect.

Definition 2.5.3 (Chromatic Right-Neighbour)**.** *The chromatic right-neighbour chord is the chord obtained by shifting every pitch in it up by one semitone, or effectively* ***sharpening*** *it. So, given a chord κ with pitches:*

$$\kappa = \{\pi_1, \pi_2, \ldots, \pi_N\}_{N \in \mathbb{N}}$$

We compactly write this:

$$\kappa \odot \sharp = \{\pi_i \odot \sharp\}_{i=1}^{N}$$

More simply,

$$\kappa \odot \sharp = \{(\pi_1 \odot \sharp), (\pi_2 \odot \sharp), \ldots, (\pi_N \odot \sharp)\}$$

2.5.3 Labelled Pitch Interval Arithmetic

Now, we have recently discussed the *Accidental-Transpositional Preservation Principle*. This means that we know how to spell many new intervals in $\mathcal{L}^*$. Now, we formalize this notion with pitch arithmetic notation. Of course, the interval math is still the same. There is nothing major changing. The only thing is that we are giving a whole set of specialized notation to drill in the different nature of this arithmetic, which is specialized to deal with note labels and ease the process of doing math with them.

Definition 2.5.4 (Pitch Product)**.** *Consider a labelled pitch $\pi \in \mathcal{L}^*$, and an interval $\iota \in \mathcal{E}_{12}$ with a frequency scalar κ. The product pitch is denoted $\pi \odot \iota$:*

$$\Pi_\iota(\pi) = \pi \odot \iota = \pi' \mid \pi' \in \mathcal{L}^*$$

which satisfies:

$$\mathit{Freq}(\pi') = \kappa \cdot \mathit{Freq}(\pi)$$

(Fifths)

From our white keys, we know that we have the chain of fifths by the furry cat sequence. So for example,

$$\Pi_\varphi(\pi) = \varphi(\pi) = \pi_7$$

We can now say some meaningful things. Suppose $\pi_k = \pi \odot q_k$. Then, we have that $\pi_7 = \varphi(\pi)$, the perfect fifth.
Perfect Fifth. Now, suppose $\mathrm{Letter}(\pi_0) \neq B$ is any letter $L \in \mathcal{L} \setminus \{B\}$. Then, due to the **tiling semitone bridge gap** occurring on B:

$$\mathrm{Letter}(\pi_0) \neq B \iff \mathrm{Acc}(\pi_7) = \mathrm{Acc}(\pi_0)$$

Equivalently, in interval notation:

$$\mathrm{Letter}(\square) \neq B \iff \mathrm{Acc}(\varphi) = \mathrm{Acc}(\square)$$

Whereas if the initial letter is B,

$$\mathrm{Letter}(\pi_0) = B \iff \mathrm{Acc}(\pi_7) = \mathrm{Acc}(\pi_0) \odot \sharp$$

Equivalently, in interval notation:

$$\mathrm{Letter}(\square) = B \iff \mathrm{Acc}(\varphi) = \mathrm{Acc}(\square) \odot \sharp$$

Note. Sufficiency of Furry Cat Theorem and the Chromatic Theorem (writing intervals by memorizing white-key tables and applying the Chromatic Algorithm) give the minimum, sufficient theoretical basis to write tonal seventh chords very easily.

Use the chromatic algorithm (i.e. the music-theory version of borrowing, like in 3rd grade addition) to find the qualities of intervals. Alternatively, can use chromatic neighbours to memorize patterns at large. For example, $A - F$ is a minor sixth implies $A^\flat - F$ is a major one. Flattening the floor or raising the ceiling augments the interval by $+1$. Raising the floor or lowering the ceiling diminishes it by -1. The accidental's effect depends on whether we are putting it on the root or on the upper note.

We know $A - F$ is a minor sixth since $F = \Lambda_6(A)$ and A is the white-key Aeolian scale.

In other words, B fifths tend to have mismatched accidentals. More colloquially, B letters have cursed shell spellings.

$$\varphi(B^\flat) = F^\natural, \quad \varphi(B^\natural) = F^\sharp, \quad \varphi(B^\sharp) = F^{\mathbf{x}}$$

Perfect Fourth. Analogously:

$$\mathrm{Letter}(\pi_0) \neq F \iff \mathrm{Acc}(\pi_5) = \mathrm{Acc}(\pi_0)$$

$$\mathrm{Letter}(\pi_0) = F \iff \mathrm{Acc}(\pi_5) = \mathrm{Acc}(\pi_0) \odot \flat$$

In the same vein of reasoning, F fourths tend to have mismatched accidentals. More colloquially, F letters have cursed fourth spellings.

$$\rho(F^\natural) = B^\flat, \quad \rho(F^\sharp) = B^\natural, \quad \rho(F^\flat) = B^{\flat\flat}$$

(**Tritones**)

So for tritones, we have:

$$\text{Letter}(\pi_0) \neq B \iff \text{Acc}(\pi_0, \pi_6) = (\natural, \flat)$$

So above, we could have just used:

$$\mathbb{A} = (\natural, \flat)$$

(**Steps: 1 + 1 = 2**)

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	B	2	w
C	D	2	w
D	E	2	w
F	G	2	w
G	A	2	w
B	C	1	h
E	F	1	h

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	$B^\flat$	1	h
C	$D^\flat$	1	h
D	$E^\flat$	1	h
F	$G^\flat$	1	h
G	$A^\flat$	1	h
B	$C^\sharp$	2	w
E	$F^\sharp$	2	w

Half-Step. We notice a pattern by C major intrinsic bias.

$$\text{Letter}(\pi) \in \{B, E\} = \overline{\textbf{Wht}}[\sharp] \iff \text{Acc}(\pi_1) = \text{Acc}(\pi_0) \odot \natural$$

So, if the letter is not black-sharp supported, it basically means we have two consecutive white keys. As such, they are both natural.

$$\text{Letter}(\pi) \in \{A, C, D, F, G\} = \overline{\textbf{Blk}}_\sharp \iff \text{Acc}(\pi_1) = \text{Acc}(\pi_0) \odot \sharp$$

Whole-Step. Analogously,

$$\text{Letter}(\pi) \in \{B, E\} = \overline{\textbf{Wht}}[\sharp] \iff \text{Acc}(\pi_2) = \text{Acc}(\pi_0) \odot \sharp$$

So, if the letter is not black-sharp supported, it basically means we have two consecutive white keys, as we've already mentioned. So, to go a whole step above, we need to go to the sharps of our white tone right-neighbour.

$$\text{Letter}(\pi) \in \{A, C, D, F, G\} = \overline{\textbf{Blk}}[\sharp] \iff \text{Acc}(\pi_2) = \text{Acc}(\pi_0) \odot \natural$$

(Thirds: 2 + 2 = 3)

Suppose $\Pi_T(\pi_0) = \pi_4$ is the major third above a pitch $\pi \in \mathcal{L}^*$. Recall the table of white thirds:

L	Λ_3	$d_3(L, \Lambda_3)$	Symbol
A	C	3	∇
B	D	3	∇
D	F	3	∇
E	G	3	∇
C	E	4	Δ
F	A	4	Δ
G	B	4	Δ

And the corresponding complementary fixes/adjustments:

L	Λ_3	$d_3(L, \Lambda_3)$	Symbol
A	$C^\sharp$	4	Δ
B	$D^\sharp$	4	Δ
D	$F^\sharp$	4	Δ
E	$G^\sharp$	4	Δ
C	$E^\flat$	3	∇
F	$A^\flat$	3	∇
G	$B^\flat$	3	∇

Major Third. We notice a pattern by **C major bias**.

$$\text{Letter}(\pi) \in \{C, F, G\} \iff \text{Acc}(\pi_4) = \text{Acc}(\pi_0) \odot \natural$$

$$\text{Letter}(\pi) \in \{A, B, D, E\} \iff \text{Acc}(\pi_4) = \text{Acc}(\pi_0) \odot \sharp$$

Minor Third. Analogously,

$$\text{Letter}(\pi) \in \{C, F, G\} \iff \text{Acc}(\pi_3) = \text{Acc}(\pi_0) \odot \flat$$

$$\text{Letter}(\pi) \in \{A, B, D, E\} \iff \text{Acc}(\pi_3) = \text{Acc}(\pi_0) \odot \natural$$

(**Accidental Family**)

Minor Thirds.

Letter	Left-Neighbors	Original	Right-Neighbours
L	$\flat \odot \kappa$	$\natural \odot \kappa$	$\sharp \odot \kappa$
A	$(A^\flat, C^\flat)$	(A, C)	$(A^\sharp, C^\sharp)$
B	$(B^\flat, D^\flat)$	(B, D)	$(B^\sharp, D^\sharp)$
C	$(C^\flat, E^{\flat\flat})$	$(C, E^\flat)$	$(C^\sharp, E)$
D	$(D^\flat, F^\flat)$	(D, F)	$(D^\sharp, F^\sharp)$
E	$(E^\flat, G^\flat)$	(E, G)	$(E^\sharp, G^\sharp)$
F	$(F^\flat, A^{\flat\flat})$	$(F, A^\flat)$	$(F^\sharp, A)$
G	$(G^\flat, B^{\flat\flat})$	$(G, B^\flat)$	$(G^\sharp, B)$

Major Thirds.

Letter	Left-Neighbors	Original	Right-Neighbours
L	$\flat \odot \kappa$	$\natural \odot \kappa$	$\sharp \odot \kappa$
A	$(A^\flat, C)$	$(A, C^\sharp)$	$(A^\sharp, C^{\sharp\sharp})$
B	$(B^\flat, D)$	$(B, D^\sharp)$	$(B^\sharp, D^{\sharp\sharp})$
C	$(C^\flat, E^\flat)$	(C, E)	$(C^\sharp, E^\sharp)$
D	$(D^\flat, F)$	$(D, F^\sharp)$	$(D^\sharp, F^{\sharp\sharp})$
E	$(E^\flat, G)$	$(E, G^\sharp)$	$(E^\sharp, G^{\sharp\sharp})$
F	$(F^\flat, A^\flat)$	(F, A)	$(F^\sharp, A^\sharp)$
G	$(G^\flat, B^\flat)$	(G, B)	$(G^\sharp, B^\sharp)$