Music of the Primes: A Constructive and Mathematical Approach to Music Theory

Prelude

(Motivations of the Book)

Story of muslims, and my pantheistic argument for God. A response to those that make music haram.

(Structure of the Book)

Nate Wells story: worst movie ever. First section of each chapter contains all the information needed. Each of the first sections, which I call the "summarized slates" are self-sufficient. Reading every first section, should hopefully in theory give you practical tips. What follows is the sections that "prove" or "construct" those shortcuts. I actually advise avoiding those until you have gained practical use of those sections. Then, gain the deeper understanding by reading through the sections carefully.

- Number systems theme and chapter one.

Also, talk about Mark Hopkins, cartoons and approachability. Color theory and Jamie pomersheim. Jamie and his ecstatic chalk and that 112 lectures where he talked about 3/2, 5/4.

Deliberate Color Theory based Choices.

Summarize each chapter. Roughly, but not exactly (due to cross-intersectionality, we cant do linear - we lose narrative!)

- Number Systems – Tuning Arithmetic
- Musical Pitch Letter-Accidental Labels - Melodic-Chromatic Arithmetic
- Tonal Group Theory– Harmonic-Tonal Arithmetic

(Notational Preliminaries)

Quick spiel on notation. I actually have known this mathematics and have been polishing and developing for at least 2 years. However, my only road block was notation. As a mathematician, absolute inambiguity was my guiding principle here. Music theory has had a bad rep due to the overabundance and liberal use of numerology to describe different, interacting things. I sought the "perfect notation" that fixes many of those problems yet at the same time motivates intuition. My mathematical statistics professor, although at times I got annoyed at his coursework, do admit he paid off his "loan" since he really wanted us to learn, and so he has prepared lectures well in advance and with notation meant to manipulate the knolwedge almost like a beam. Thanks Weiner.

0.1 Mathematical Preliminaries

Basic set theory. This is optional, but can enhance your understanding. It is definitely unnecessary but I do recommend taking a stab at it. If it is not your cup of tea, I understand – try to make the most out of this book. You certainly can understand 100% of the material without any of the abstract math. However, I will say, you will gain aesthetic satsifaction from the beauty the math holds.

Set Theory Primer.

Equivalency Relations.

$\mathbb{Z}/12\mathbb{Z}$ **Arithmetic - Clock Math.**

0.2 Number Systems

Before we begin, we review the three basic number systems in set-theoretic notation: the natural numbers, the integers, and the rational numbers.

The natural numbers, denoted as $\mathbb{N}$ are the counting numbers, including 0. We can also call them whole numbers.

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

The integers, denoted as $\mathbb{Z}$ are defined as the natural numbers, along with their inverses (negative counterparts).

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

Note that the non-zero integers, which we denote $\mathbb{Z}_{\neq 0}$ is defined as the set $\mathbb{Z} \setminus \{0\}$:

$$\mathbb{Z}_{\neq 0} = \{\dots, -2, -1, 1, 2, \dots\}$$

The rational numbers, denoted as $\mathbb{Q}$ are defined as the set of possible ratios formed by dividing the **numerator**, an integer ($n \in \mathbb{Z}$), by the **denominator**, a non-zero integer ($d \in \mathbb{Z}_{\neq 0}$).

$$\mathbb{Q} = \left\{ \frac{n}{d} = nd^{-1} \mid n \in \mathbb{Z}, d \in \mathbb{Z}_{\neq 0} \right\}$$

(**Notational Breakdown**)

Greek/Latin Letters.

L, Λ from a root letter to a letter derived from it. π, Π from a pitch to a derived pitch letter. same conecpt.

Accidentals (Unaccidentalized vs Accidentalized, $\alpha \in \mathcal{A}$, L^*, $L \in \mathcal{L}$) Lydianic Dyadic Stacks (Collierian Path) Shell Theory

(**Notational Summary**)

k	q_k	Interval
0	$\square_0$	Unison
1	h	Semitone
2	w	Whole Tone
3	t	Minor 3^{rd}
4	T	Major 3^{rd}
5	ρ	Perfect 4^{th}
6	$\pitchfork$	Tritone
7	φ	Perfect 5^{th}
8	s	Minor 6^{th}
9	S	Major 6^{th}
10	v	Minor 7^{th}
11	V	Major 7^{th}
$12k$	$\square_k$	Octave

Octave - Octagons too many sides. 4 related to and 8. So $\square$, since 4 sides. Actually $8 = 2 \cdot 4 = 2 \cdot 2^2$ Unison is the root or tonic. Special case - 0 octaves. It is called an identity element, like 0 in addition and 1 in multiplication. The doing nothing operators in math are actually quite important.

1,2 - Dont reinvent the wheel principle. The absolute ubiquity of using w and h in music leads to us naturally adopting them. Also, gives special status to the second/step intervals. However, there exists an alternative labeling using $\{h, w\} = \{u, U\}$ which we do in later chapters. This will be clarified later. We use $\mathbb{U}$ since 2 = deux in French.

$\mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}$, we pair the consonances sixths, thirds, an dissonances seconds and sevenths together. So neat!

The perfect intervals use **Greek** letters: ρ and φ for r in fourth and f in fifth - the fifth takes precedent, even if fourth starts with the same letter, since its super important. I like to be neutral but the fifth is nicer. Also two f's is a better explanation if you still mad.

The breaking of the perfect intervals, tritone, A4, d5, is given a pitchwork since it contains 3 spokes, and the tritone is called so because tones were whole tones and the tritone is 3 whole tones as $6 = 2 \cdot 3 \iff \pitchfork = w^3$.

To summarize: the tonal intervals with qualities use **Latin** letters as we will use them in our self-developed tonal theory. So, we have $\mathbb{S}, \mathbb{T}, \mathbb{V}$. Since seconds are special, we dont reinvent the wheel and use h, w. However, to complete the theory, we have the super convenient extension or renotation of the seconds using $\mathbb{U}$.

Perfect intervals use non-latin letters, $\square$, ρ, φ. The breaking of the generators, $\pitchfork = \rho^{+} = \varphi^{\circ}$ is also non-latin, alluding to the definition of tritone- 3 spokes, 3 (whole) tones.

Principle of Non-Confounding Notation/Labels. Shit on interval pitch set class for using numbers - to talk about intervals, which use numbers, and different capitalizations of M, m. WTF. So, we use letters. However, since the musical alphabet uses $\mathcal{L} = \{A, B, \ldots, F, G\}$, we solve the problem of non-confounding by just looking beyond them and making sure we don't reuse the same letter ever again. This applies to the whole book! We find some neat solutions which actually pair up quite nicely, the imperfect/qualitied intervals (which could be moons or suns) are actually all consecutive: $\mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}$. Even better, we actually have within those consecutive ones, relations that are consecutive.

Consonances are 3, 6 = $\mathbb{S}, \mathbb{T}$

Dissonances are 2, 7 = $\mathbb{U}, \mathbb{V}$

Using this new toolkit, we explain away various confusions.

Warning (Example of Confusion)**.** *Two thirds of alternating quality add to a fifth, they make up major and minor chords, why does two thirds make a fifth and not a sixth? Well, where we end in the beginning, we start again. So, the second third is actually like a second due to double counting. Also, neat!*

$$t + T = T + t = \varphi$$

“And say Universe, endow me your knowledge.”

Su’rat Taha – (20 : 114)

«وقل ربي زدني علمًا »
[طه: ١١٤]

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Mathematical Genesis of Pitch Tuning

1.1 Chapter Summary

(**Summary: Tuning Arithmetic**)

1.2 What is a Note?

1.2.1 Notes = Tones (Ψ)

Definition 1.2.1 (Tone)**.** *A tone τ is a vibration through some physical medium percieved as a **musical note**. Depending on the accousticity of the physical-material constraints, a note is more prominent if its harmonius and produces some overtones. It is characterized by a frequency:*

$$Freq(\tau) = f_0$$

Where f_0 is some frequency in a range of pitches, which we give the fancy name **pitch continuum** and label it Ψ. To summarize, we have a tone τ whose freuqency f_0 lives in Ψ. We write this:

$$\tau : \mathrm{Freq}(\tau) = f_0 \mid f_0 \in \Psi$$

Of course, us humans within through our *cochleas* and passing through our *eardrums*. These are ***nature's constraints*** which impose natural bounds on where our hearing range cuts off, in both the ***low and high ranges***. So, to practically account for this, we fix a lowest frequency L and a highest frequency H; togther, they define the pitch continuum Ψ as the continous range of frequencies between these two bounds.

$$\Psi = [L, H] = \{f \in \Psi' \mid L \leq f \leq H\}$$

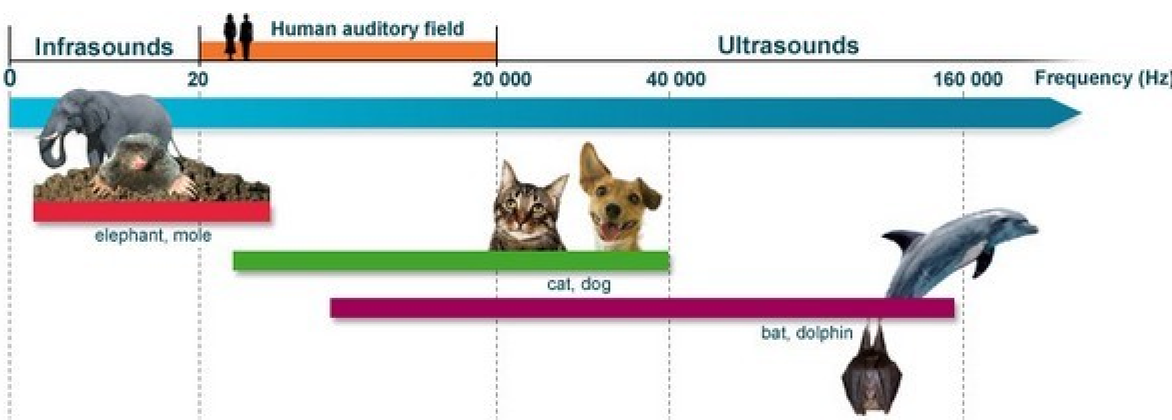
Note (Theoretical Constructs)**.** *where Ψ' is just some theoretical construct of all the possible range of pitches, effectively equivalent to the space of positive (non-zero) rational numbers $\mathbb{Q}^+$ that represent the space of frequency scalars $\imath \in \mathbb{Q}^+$ tuned to some central frequency f_0. So, we can formally say Ψ is chosen to be our restricted subset of tones from Ψ' which we write:*

$$\Psi \subset \Psi'$$

Effectively speaking, a practical range, which we will actually call Ψ hereonafter, is this particular, arbitrary example.

Example (Selected Pitch Continuum)**.** *Let $L = 20Hz$, $H = 20,000Hz$. So, our pitch continnum is $\Psi = [20Hz, 20kHz]$, the range of tones in the range:*

$$\Psi = \{f : 20Hz \leq f \leq 20,000Hz\}$$



1.2.2 The Harmonic Series ($\mathbb{N}$)

The story of music begins with the discussion of pitch. A pitch, in simple terms, is a a vibration at a fixed frequency. Let's say we have a pitch at a frequency

$f = 440$Hz. This means that we have 440 vibrations per second, which we percieve through cycles of repeated compression and decompression of air every $\frac{1}{440}$ seconds. As such, the simplest repestentation of a pitch is a perfect sinusoid. So, the frequnecy $f = 440$Hz is essentially an even beat, completing 440 cycles in 1 second.

So, we could model a tone at that frequency to have displacement (or compression intensity) modeled by the sinusoidal equation:

$$d(t) = A \sin(\pi f t) = A \sin(440\pi t)$$

This means that between $t = 0$ and $t = 1$, which is the span of 1 second, the wave completes 440 oscillations at some amplitude or intensity A. This is since $\sin(\pi n) = 0$ for any $n \in \mathbb{N}$.

Definition 1.2.2 (Intervals)**.** *An interval $\iota \in \mathbb{R}^+$ is a positive real number scalar which takes a pitch τ and returns another pitch $\tau\iota$ such that:*

$$Freq(\tau\iota) = \iota \cdot Freq(\tau)$$

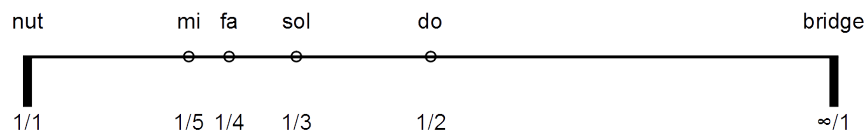
If $\iota < 1$, we call it descending; if $\iota > 1$, we call it ascending; otherwise, we call $\iota = 1$ a unison.

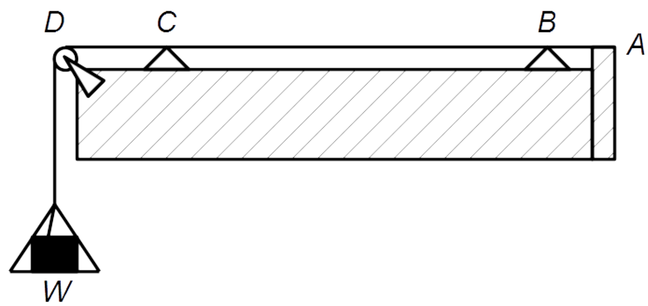
That being said, let us consider taking the superposition of another frequency. If we have a frequency f, what other frequency f' would sound **consonant/harmonious** with it? If we consider the notion of "constructive interference" as a physical guage, then we try to align 2 beats per beat, so the waves meet every other oscillation.

This, of course translates to just taking double the frequency, so letting $f = 440$Hz, we would take $f' = 880$Hz.

Storytime 1 (Pythagoras' Hammers)**.** *This relationship is rumored to have been discovered by Pythagoras, when while walking one day, he heard two hammers, one double the length of the other being struck at the same time. He found that they were consonant, since length translate through the mathematics of the physics to frequency by computations involving the wavelength. In physics, $f = v/\lambda$.*

(**Pythaogras' Monochord**)





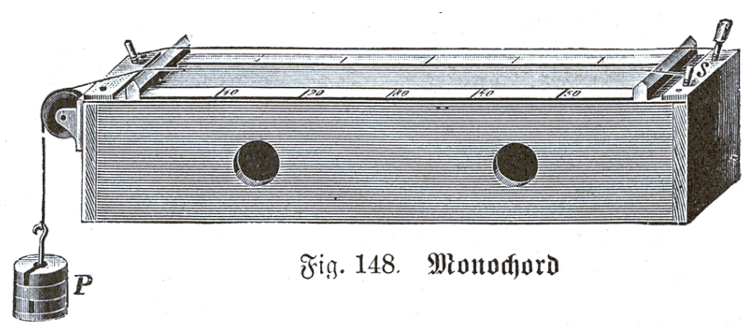


Fig. 148. Monochord

Extrapolating this principle, if we take some frequency f, then we know that the frequency $f' = nf$ for some $n \in \mathbb{N}$ has **constructive interference** every n^{th} oscillation. This extrapolation naturally leads us to defining the harmonic series:

Definition 1.2.3 (Harmonic Series)**.** *The harmonic series of τ, denoted $\mathcal{H}_\tau$ is a series of frequencies, beginning with the fundamental frequency τ. It includes its* ***harmonics****, which are the resulting series of derived frequencies of its (positive) whole number multiples, which we denote $\mathbf{h}_n(\tau) = n\tau \in \Psi$, where $n \in \mathbb{N}^+$. So, we define the* ***harmonic series*** *of τ to be the set of frequencies:*

$$\mathcal{H}_\tau = \Psi \cap \{\mathbf{h}_n(\tau) = n\tau\}_{n \in \mathbb{N}^+}$$

More simply put,

$$\mathcal{H}_\tau = \{\tau, 2\tau, 3\tau, \dots\}$$

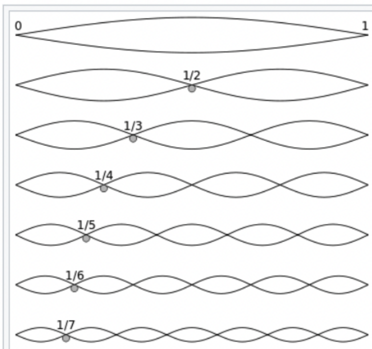
Analogously, the **overtone series** of τ are the set of harmonics of τ, disincluding the fundamental (hence the name, "overtone"). We could also think of the overtone series as a phased version the harmonic series, starting from the first harmonic instead. As such, the overtone series is defined:

$$\mathcal{O}_\tau = \Psi \cap \{\mathbf{h}_{n+1}(\tau) = (n+1)\,\tau\}_{n \in \mathbb{N}^+}$$

More simply put,

$$\mathcal{O}_\tau = \{2\tau, 3\tau, 4\tau, \dots\}$$

This implies that the 1^{st} overtone is equal to the second harmonic, 2τ. Likewise, the 2^{nd} overtone is 3τ, etc. That being said, we now discuss a mathematical bridge between the world of frequencies in physics and consonance in music.



The fundamental and the first six overtones of a vibrating string. The mathematics of wave propagation in crystalline solids consists of treating the harmonics as an ideal Fourier series of sinusoidal density fluctuations (or atomic displacement waves).

Note (Small Numbers & Consonant Constructive Interference)**.** *We know that two waves produce a nice consonant beating pattern, called a* ***polyrhythm*** *(more later), when two waves share quite a few number of "beats". In physics, this would correspond to constructive interference. The consonance is thus derived from the consonant rhythmic patterns between the two pulses. Essentially, this means that the harmonic series is a key to finding consonace, where lower harmonics tend to be more prominent and significant.*

That being said, the fundamental itself is perfectly consonant, but it is a trivial relation. So, we search for relations that increasingly non-trivial but at the same time, less consonant due to increasing percieved polyrhythm complexity. As such, the overtones impose a natural ranking of intervals by complexity. So far, we have talked about the story of the number 2, as with Pythagoras and his hammers.

1.2.3 Octaves & Integers ($\mathbb{Z}$)

The consonance of the first overtone is so prominent, that it is the basis of two notes sounding as the **same note** if they are some whole number of octaves apart! Now, before we continue, we will define the first interval: the octave.

Definition 1.2.4 (Octave)**.** *Consider a note τ. Then, the note an octave above τ is denoted $\square_1 \tau = 2^1\tau$. In fact, we may generalize, and say the note $\square_k\tau$ is k octaves above τ where:*

$$\square_k \tau = 2^k\tau$$

Note that this implicitly gives us a neat convention, where we implicitly have a notion of 0 octaves:

$$\square_0 \tau = 2^0\tau = 1\tau = \tau$$

As such, $\square_k$ gives us notes that are k octave(s) above or below τ, where $k \in \mathbb{Z}$ is any integer. If k is negative, then it gives us a note octaves below our original. We could also just say:

$$\square_{-k} = 2^{-k} = \left(2^{-1}\right)^k = \left(\frac{1}{2}\right)^k$$

Note (Pitch Equivalence)**.** *The octave is universally, with no exception, seen as some form of pitch equivalence. This forms an equivalence class on any pitch $\pi \in \Psi$, which includes every octave note above or under it.*

$$[\pi]_\sim = \{\square_k\pi \in \Psi\}_{k\in\mathbb{Z}}$$

Of course, our ears are constrained to only hear a finite range of pitches, so constrict π to some finite pitch continuum $\Psi = [\tau_0, \tau_1]$.

Definition 1.2.5 (Octave Equivalency)**.** *The octave equivalency relation $\sim_{8v}$ partitions a set of notes in a pitch continuum Ψ into disjoint bins of what we consider the same note. For instance, in our space, we have about 6-7 octaves for every note on a grand piano. We consider two pitches equal when:*

$$\pi \sim_{8v} \pi' \iff \pi' = 2^K \text{ for some } K \in \mathbb{Z}$$

So, the disjoint bins formed by these pitches are notated as a set $[\pi]_{\sim_{8v}}$ where:

$$[\pi]_{\sim_{8v}} = \{\square_k\pi \in \Psi\}_{k\in\mathbb{Z}}$$

(Scales)

Now, we define a musical scale. In Solfege, we have this in the form of a (Do-Do) Sonority.

Definition 1.2.6 (Scales)**.** *A scale is a finite set of intervals that all lie within an octave. Suppose we have a scale $\mathcal{S}$ tuned about tuning frequency τ. Then, the scale $\mathcal{S}$ is characterized by $\mathbb{I}$, a finite set of N intervals $q \in \mathbb{Q}^+$*

$$\mathbb{I} = \{q_i \in [1, 2]_{\mathbb{Q}^+} \mid i \in \mathbb{N}\}_{i=1}^{N}$$

This means the scale has the notes:

$$\mathcal{S} = \tau \times \mathbb{I} = \left\{\tau q_i \mid q_i \in \mathbb{I}\right\}_{i=1}^{N}$$

*By defining scales in this way, we are able to generalize analysis for any τ and focus on the dynamic part, being $\mathbb{I}$. As such, we interpret tuning systems as **parameterized scales**.*

That being said, the reason scales are brought up in this subsection is because scales repeat at the octave. This of course is due to the definition, since scales partition an octave, it means that when we go up a scale and reach its end (the octave-above note), it starts all over again.

(Major and Minor Scales)

Definition 1.2.7 (Solfege)**.** *Do, Re...*

Major Scale. Do, Re..

So, we have a 7-note scale. Here, the difference between (Do-Re) is one interval, the second... etc. Each interval lies in $[1, 2]_{\mathbb{Q}}$.

Minor Scale. La, Ti..

1.3 Making New Friends: Intervals

1.3.1 Frequency Scaling & Ratios ($\mathbb{Q}$)

(**Consonances and Overtones**)

That being said, we now have a notion of two notes being the "same". Every note is perfectly consonant with itself, but we are now stuck with just one note. This is boring; we can't make much music with one note. Naturally, this leads us to ask the question, if 2τ, the first overtone is consonant, what else is the next best consonance? The answer is the next overtone! This corresponds to the frequency 3τ. Recalling our dialogue connecting overtones and consonance, this puts our sight on the second overtone/third harmonic: $\mathbf{h}_3(\tau) = 3\tau$. This turns out to yield a note equivalent to the **perfect fifth**! Specifically, it is the note one octave above the perfect fifth.

Definition 1.3.1 (Scale-Adjusted Intervals)**.** *Consider an post-octaval, ascending interval $\imath \in [2, \infty)_{\mathbb{Q}}$. Then, the corresponding scale-adjusted interval, which we denote $\tilde{\imath} \in (1, 2]_{\mathbb{Q}}$ is the interval transposed k octaves such that it fits in a scale. In other words, octave-transposed to a corresponding ratio in* $(1, 2]$. *Mathametically, it is the k-octave shifted version of $\imath$, where k satisfies:*

$$1 < (2^k) \cdot \imath < 2$$

As we will formally discuss later, a scale (tuned about τ) is essentially a set of intervals that yield frequencies between some fundamental note frequency τ and 2τ (one octave above). The only takeaway here is that we generally prefer intervals to have frequency ratios between 1 and 2. So, we defined scale-adjustment of overtones.

Definition 1.3.2 (Scale-Adjusted Harmonics)**.** *Consider a harmonic $\mathbf{h}_n = n$, where $n \in \mathbb{N}_{\geq 2}$. Then, the scale-adjusted harmonic, denoted $\mathbb{H}_n \in [1, 2]_{\mathbb{Q}}$, is the n^{th} harmonic $\mathbf{h}_n$ transposed k octaves such that:*

$$1 < (2^k) \cdot n \leq 2$$

For a frequency τ, it is the k-octave shifted version of $n\tau = \mathbf{h}_n(\tau)$, denoted $\mathbb{H}_n(\tau) = \square_k \mathbf{h}_n(\tau)$ where $k \in \mathbb{Z}$ is some integer that satisfies the above equation so that we may have,

$$\tau < \square_k \mathbf{h}_n(\tau) \leq 2\tau$$

so the new pitch $\mathbb{H}_n$ is in a scale of τ for any $n \in \mathbb{N}_{\geq 2}$:

$$\tau < \mathbb{H}_n(\tau) \leq 2\tau$$

Note (Notation of $\mathbb{H}_n$)**.** *By the way this function was defined and our notation is laid out, one could either think of $\mathbb{H}_n$ as a number between (1,2], or a function of τ that returns a pitch in $(\tau, 2\tau]$!*

So, if we were to take the third harmonic 3τ, the solution to finding the corresponding note that would be in a scale of τ. To start off, we have to do **something** since $k = 1$ yields 3, and clearly, $3 > 2$. So, we transpose it an octave down by dividing its frequency by 2 (or multplifying by $\frac{1}{2}$) to obtain $\frac{3}{2}\tau$. Since $1 < \frac{3}{2} \leq 2$, the note $\frac{3}{2}\tau$ could be in a scale of τ. So, $\mathbb{H}_3 = \frac{3}{2}$. This is precisely the definition of the justonic perfect fifth! It is so special, we will denote it with a special symbol φ.

Definition 1.3.3 (Perfect Fifth)**.** *The perfect fifth of τ is the interval derived from the scale-adjusted third harmonic, $\mathbf{h}_3(\tau) = 3\tau$, which we find to be $\mathbb{H}_3(\tau) = \square_{-1}(3\tau) = \frac{3}{2}\tau$. We denote the perfect fifth φ:*

$$\varphi(\tau) = \frac{3}{2}\tau$$

That being said, we continue to traverse the overtone series. The third overtone, 4τ is not as special as the first two. This follows since $4\tau = 2^2\tau = \square_2(\tau)$, which means it is the same note as τ, two octaves higher. Equivalently, it is the first overtone of its own first overtone – showing a neat recursive property!

Now, we won't go on forever, and in fact will stop here – consider the fourth overtone, 5τ. Transposing it down to fit in a scale, the frequency $\frac{5}{4}\tau$ sounds at a **major third** above τ, which is the next most consonant interval, after the **perfect fifth**. So, we define the major third to be as follows:

Definition 1.3.4 (Major Third)**.** *The major third of τ is the scale-adjusted fifth harmonic (or fourth overtone), $\mathbb{H}_5$. We find that $\mathbb{H}_5 = \square_{-2}(5) = \frac{5}{4}$. We denote the major third T, meaning:*

$$T(\tau) = \frac{5}{4}\tau$$

Below, we have a table describing the first 10 scale-adjusted harmonics.

n	$\mathbf{h}_n$	$\mathbb{H}_n$	Ratio	k
2	$\square_1$	$\square$	2	0
3	$(\square_1) \cdot \varphi$	φ	$\frac{3}{2}$	-1
4	$\square_2$	$\square$	2	-2
5	$(\square_2) \cdot T$	T	$\frac{5}{4}$	-2
6	$(\square_2) \cdot \varphi$	φ	$\frac{3}{2}$	-2
7	$(\square_2) \cdot \mathbb{H}_7$	$\mathbb{H}_7$	$\frac{7}{4}$	-2
8	$\square_3$	$\square$	2	-3
9	$(\square_3) \cdot w$	w	$\frac{9}{8}$	-3
10	$(\square_3) \cdot T$	T	$\frac{5}{4}$	-3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Notice, from the table above, we discover the new relation:

$$\mathbb{H}_n = \mathbb{H}_{(2^k)n}$$

for any $n \in \mathbb{N}^+$ and $k \in \mathbb{N}^+$.

Example (Major Third)**.** *The major third T is represented by the following harmonic subseries:*

$$T = \mathbb{H}_5 \sim_{8v} \{\square_k \mathbb{H}_5\}_{k \in \mathbb{N}^+} = \{\mathbb{H}_{(2^k)5}\}_{k \in \mathbb{N}^+} = \{\mathbb{H}_5, \mathbb{H}_{10}, \mathbb{H}_{20}, \mathbb{H}_{40}, \mathbb{H}_{80}, \dots\}$$

corresponding to the subset A_5:

$$A_{P5} = \{5 \cdot 2^k \mid k \in \mathbb{N}^+\} = \{5, 10, 20, 40, 80, \dots\}$$

Example (Perfect Fifth)**.** *The perfect fifth φ is represented by the following harmonic subseries:*

$$T = \mathbb{H}_3 \sim_{8v} \{\square_k \mathbb{H}_3\}_{k \in \mathbb{N}^+} = \{\mathbb{H}_{(2^k)3}\}_{k \in \mathbb{N}^+} = \{\mathbb{H}_3, \mathbb{H}_6, \mathbb{H}_{12}, \mathbb{H}_{24}, \mathbb{H}_{48}, \dots\}$$

corresponding to the subset A_3:

$$A_{M3} = \{3 \cdot 2^k \mid k \in \mathbb{N}^+\} = \{3, 6, 12, 24, 48, \dots\}$$

That being said, we now have the tools to motivate the **major triad**.

1.3.2 The Major Chord ($\mathbb{P}$)

(Major Triad)

Define the set of unique intervals generated by the harmonic series:

$$\mathbb{H}_{\mathbb{P}} = \{\mathbb{H}_2, \mathbb{H}_3, \mathbb{H}_5, \mathbb{H}_7, \mathbb{H}_{11}, \dots\}$$

We call collectively the resulting justonic intervals **the justonic major triad**:

$$\begin{aligned} \Delta[\tau] &= \{a\tau \,:\, a \in \mathbb{I}_\Delta\} \\ &= \{\tau, (5/4)\,\tau, (3/2)\,\tau, 2\tau\} \end{aligned}$$

So, this A_Δ harmonically subseries, as simple as it is, induces $\mathbb{I}_\Delta$, which all musicians concur to be the bedrock of consonance: **the justly-intonated major triad**, as it naturally appears in the harmonic series. It corresponds to the set of sonorities (Do, Mi, So, Do). This means the justonic major triad is composed of the justonic intervals:

$$\mathbb{I}_\Delta = \left\{1, \frac{5}{4}, \frac{3}{2}, 2\right\}$$

Note (Prime Numbers)**.** *It is no surprise then, to find that the major triad is equivalent to finding the scale-adjusted harmonics of the subseries corresponding to the 3 lowest prime numbers:* $\{2, 3, 5\}$*! Since these numbers are the building blocks, it shows a neat balance between the complexity of a polyrhythm and the order it appears in the prime numbers.*

Common term	Example name	Hz	Multiple of fundamental	Ratio of within octave	Cents within octave
Fundamental	A_2	110	$1x$	$1/1 = 1x$	0
Octave	A_3	220	$2x$	$2/1 = 2x$	1200
				$2/2 = 1x$	0
Perfect Fifth	E_4	330	$3x$	$3/2 = 1.5x$	702
Octave	A_4	440	$4x$	$4/2 = 2x$	1200
				$4/4 = 1x$	0
Major Third	C♯$_5$	550	$5x$	$5/4 = 1.25x$	386
Perfect Fifth	E_5	660	$6x$	$6/4 = 1.5x$	702
Harmonic seventh	G_5	770	$7x$	$7/4 = 1.75x$	969
Octave	A_5	880	$8x$	$8/4 = 2x$	1200
				$8/8 = 1x$	0

1.4 Modern Tuning: 12-TET

1.4.1 A Curious Approximation ($\mathbb{R}$)

That being said, let's go back to the major triad, including an octave note. Recall that it was given by the justonic interval ratios:

$$\mathbb{I}_\Delta = \left\{1, \frac{5}{4}, \frac{3}{2}, 2\right\} = \{1, 1.25, 1.5, 2\}$$

Consider the following two curious approximations:

$$2^{\frac{4}{12}} = (\sqrt[3]{2}) = 1.2599.. \approx 1.25 = \frac{5}{4} = T$$

$$2^{\frac{7}{12}} = (\sqrt[12]{2})^7 = 1.4983.. \approx 1.5 = \frac{3}{2} = \varphi$$

Putting those together with the fact that $2^0 = 1$, this means:

$$\{2^0, 2^{4/12}, 2^{7/12}, 2^1\} = \{1, 1.2599..., 1.498..., 2\} \approx \mathbb{I}_\Delta$$

So, curiously enough, taking 2 to the powers of some fractions yield good approximations for these important consonant intervals. Keep in mind, 2 is the octave multiplier, so these fractions seem to be related to correspond to the intervals of inside some scale. This follows since for any base a and fraction $k \in (0, 1]$, we have $a^k \leq a$.

As it turns out, those numbers, 4 and 7, turn out to be the number of **semitones** that yield our modern, tempered versions of these intervals! This whole time, we never really talked about the difference between our musical system and these perfect ratios. For now, we will gloss over this caveat of tuning theory.

To make sure every note is consonant with itself, every octave needs to be in-tune. So, sacrificing the mathematical purity of our interval ratios, we obtain the wonderful 12-TET system, which does a great job. The secret lies in solving the equation that defines the semitone or half-tone, which we call h:

$$h^{12} = 2 \implies h = \sqrt[12]{2} = 2^{1/12}$$

Some may be familiar with the irrational numbers, most being exposed to it the first time when solving for the hypotenuse of a isoceles right-triangle with side length 1, which is $2^{\frac{1}{2}}$, or more commonly, $\sqrt{2}$.

$$x = \sqrt{2} = 2^{\frac{1}{2}} \implies x^2 = (\sqrt{2})^2 = (2^{\frac{1}{2}})^2 = 2$$

Storytime 2 (The Bean-Hating Cultist)**.** *In Pythagoras, the cultist bean-fearer who died in fields of beans, drowned poor Hypeices only for saying irrational numbers were alright. Pythagoras really didn't like the fact that* $\sqrt{2}$*, the hypotenuse of a isoceles right-triangle with side length* 1*, was not expressible as a ratio of two numbers:*

$$\sqrt{2} \neq \frac{a}{b} \mid a, b \in \mathbb{Z}_{\neq 0}$$

This is the origin of our half-step or semitone interval scalar, which we call $q_1 = 2^{1/12}$. As such, 12 equal semitones multiply to obtain an octave:

$$(q_1)^{12} = (2^{1/12})^{12} = 2^{12/12} = 2^1 = 2$$

That being said, we now define the general N-TET system.

Definition 1.4.1 (N-Tone Equal Temperament)**.** *The N-Tone Equal Temperament system, abbreviated N-TET, is the system corresponding to subdividing the octave into N equal components. As such, an N-TET system has the interval set* $\mathbb{I} = \mathcal{E}_{12}$ *where:*

$$\mathcal{E}_N = \{q_k = 2^{k/N} \mid k = 0, 1, \ldots, N-1\}$$

What should the primary takeaway be? The 12-TET system $\mathcal{E}_{12}$ contains an approximation for every interval in our beloved justonic major triad: $\mathbb{I}_\Delta = \{1, \frac{5}{4}, \frac{3}{2}, 2\}$. Remember, these correspond to the scale-adjusted harmonics in the prime subset $\{2, 3, 5\}$.

Now, we have accquired notation to abbreviate our own tuning system! Namely, our tuning system is a pitch continuum Ψ tuned to the **A Chromatic Scale**. Specifically, it is tuned to τ = A4 = 440Hz along with an interval set $\mathcal{E}_{12}$. Consider the table at the end of the section, where we give every 12-TET interval a special symbol, some of which already been seen. That being said, to drive home the fact that our tuning system is really just the **A Chromatic Scale**, consider the neat graphic below.

Octave	0	1	2	3	4	5	6	7	8	9	10
C	-57	-45	-33	-21	-9	3	15	27	39	51	63
C♯/ D♭	-56	-44	-32	-20	-8	4	16	28	40	52	64
D	-55	-43	-31	-19	-7	5	17	29	41	53	65
D# / Eb	-54	-42	-30	-18	-6	6	18	30	42	54	66
E	-53	-41	-29	-17	-5	7	19	31	43	55	67
F	-52	-40	-28	-16	-4	8	20	32	44	56	68
F# / Gb	-51	-39	-27	-15	-3	9	21	33	45	57	69
G	-50	-38	-26	-14	-2	10	22	34	46	58	70
G# Ab	-49	-37	-25	-13	-1	11	23	35	47	59	71
A	-48	-36	-24	-12	0	12	24	36	48	60	72
A# / Bb	-47	-35	-23	-11	1	13	25	37	49	61	73
B	-46	-34	-22	-10	2	14	26	38	50	62	74

Now, we inspect a specific interval in $\mathcal{E}_{12}$, the 6-semitone interval $q_6 = \pitchfork$.

Definition 1.4.2 (Tritone)**.** *The interval $q_6 = \pitchfork$ is the tritone, defined as the frequency scalar $q_6 = \sqrt{2}$. This follows by the definition,*

$$q_6 = 2^{\frac{6}{12}} = 2^{\frac{1}{2}} = \sqrt{2}$$

Note that the $\sqrt{2}$ has some special properties. For instance,

$$\sqrt{2} = \frac{2}{\sqrt{2}} \iff \pitchfork = \frac{\square_1}{\pitchfork}$$

Also, since frequencies are in a logarithmic scale (i.e. semitone addition turns into frequency multiplication), we can interpret the $\sqrt{2}$, and accordingly the tritone, as the **logarithmic midpoint of the octave**. This can be demostrated as being the midpoint to the range $[0, 1]$, and taking the logarithmic midpoint to be the k halfway in $[0, 1]$ so the interval is $2^k = 2^{\frac{1}{2}}$.

The **geometric midpoint** is $1.5 = \frac{3}{2}$, or the perfect fifth, curiously enough!

The tritone is special because it is an irrational number that we are tuning to. Those whole time, we've been trying to approximate **towards simple ratios** and **not away** from them! Consider our major triad components:

$$T = \frac{5}{4} \quad \varphi = \frac{3}{2} \quad \square_1 = \frac{2}{1}$$

Note (The Tritone is Dissonant)**.** *The tritone is dissonant. This is connected to the fact that there exists no clear "best" simple ratio* $r \in \mathbb{Q}$ *that approximates* q_6 *best. This follows since irrational numbers, by definition, are a non-terminating sequence, so they are surrounded by infinitely many approximating rational numbers. This is in contrast to* $q_7 \sim \frac{3}{2}$.

Here is the set of intervals in $\mathbb{Z}_{12} \cong \mathcal{E}_{12}$

k	q_k	Interval
0	$\Box_0$	Unison
1	h	Semitone
2	w	Whole Tone
3	t	Minor 3^{rd}
4	T	Major 3^{rd}
5	ρ	Perfect 4^{th}
6	$\pitchfork$	Tritone
7	φ	Perfect 5^{th}
8	s	Minor 6^{th}
9	S	Major 6^{th}
10	v	Minor 7^{th}
11	V	Major 7^{th}
$12k$	$\Box_k$	Octave

1.4.2 Justonic-Approximation Connection ($\mathbb{R} \supset \mathbb{Q}$)

Now, the question begs, how are our irrational intervals $q_k \in \mathcal{E}_{12}$ in $\mathbb{R}$ related to the justonic ratios in $\mathbb{Q}$? Well, first we observe there is an assymetric inclusion relation since $\mathbb{Q} \subset \mathbb{R}$. So, we motivate a notion of closeness, up to some threshold we call ε.

Definition 1.4.3 (ε-Equivalence)**.** *Let $\varepsilon \in \mathbb{R}^+$ be a small positive real number, which we call the error threshold. Suppose we have real numbers $x, x' \in \mathbb{R}$. Then, x, x' are ε-equivalent if and only if their absolute difference doesn't exceed ε. In other words,*

$$x \sim_\varepsilon x' \iff |x - x'| < \varepsilon$$

That being said, now we have a way to explicitly quantify an equivalence between our justonic ratios from our scale-adjusted harmonics and their 12-TET interval approximations:

$$q_4 \sim_\varepsilon T \mid \varepsilon \leq 0.05$$

$$q_7 \sim_\varepsilon \varphi \mid \varepsilon \leq 0.05$$

So, if we fix corresponding some set of 11 justonic ratios $r_k \in \mathbb{Q}$ that each approximate $q_k \in \mathbb{R}$, $\mathbb{I}_\mathcal{J} = \{r_k \in \mathbb{Q}^+\}_{k \in \mathbb{N}_{11}}$, then we could say:

$$\mathcal{E}_{12} \sim_\varepsilon \mathbb{I}_\mathcal{J} \iff \forall k \in \mathbb{N}_{11} : r_k \sim_\varepsilon q_k$$

Or in other words, a justonic interval set $\mathbb{I}_\mathcal{J}$ is an ε-apprxomiate of $\mathcal{E}_{12}$ is $r_k \sim_\varepsilon q_k$ for all $k \in \{0, 1, 2, \dots, 11\}_\mathbb{N}$ Note that we say 11 since $q_0 = 1$ is just the fundamental frequency or a unison. We also assume octaves are tuned perfectly as $q_{12} = 2$.

1.4.3 Other Notable Justonic Intervals

Definition 1.4.4 (Major Second)**.** *The major second, or whole tone above τ is the scale-adjusted ninth harmonic (or eigth overtone), $\mathbb{H}_9$. We find that $\mathbb{H}_9 = \square_{-3}(9) = \frac{9}{8}$. We denote the major second w, meaning:*

$$w(\tau) = \frac{9}{8}\tau$$

In 12-TET, the major second is approximated by 2 semitones, the interval $q_2 = w$, which we call a whole tone:

$$q_2 = 2^{\frac{2}{12}} = \sqrt[6]{2} = 1.1224.. \approx 1.125 = \frac{9}{8}$$

Note (Whole Tones & Fifths)**.** *Note that there is a special relationship between the major second w and the perfect fifth φ. Namely, this derives from the fact that $3^2 = 9$, where we notice:*

$$\mathbb{H}_9 = \frac{9}{8} = \frac{1}{2} \cdot \left(\frac{3}{2}\right)^2 = 2^{-1}(\mathbb{H}_3)^2 = \square_{-1}\varphi^2$$

Or in summary,

$$\mathbb{H}_9 = \square_{-1}(\mathbb{H}_3)^2 \iff \mathbb{H}_9 \sim_{8v} (\mathbb{H}_3)^2$$

In general, this relationship is written (in future chapters):

$$w = \varphi^2$$

Definition 1.4.5 (Perfect Fourth)**.** *The perfect fourth of τ is the first interval we won't define as a scale adjusted harmonic. Rather, the perfect fourth, denoted ρ is defined as the interval between $\mathbf{h}_3 = \square_1\varphi$ and $\mathbf{h}_4 = \square_2$. We find that $\rho = \frac{\mathbf{h}_4}{\mathbf{h}_3} = \frac{4}{3}$. As such, we highlight the special relation:*

$$\rho = \frac{\square_2}{\square_1\varphi} = \frac{\square_1}{\square_0\varphi} = \frac{\square_1}{\varphi} \implies \rho\varphi = \square_1$$

So, given τ, we denote the perfect fourth $\rho(\tau)$, meaning:

$$\rho(\tau) = \frac{4}{3}\tau$$

In 12-TET, the perfect fourth is approximated by 5 semitones, the interval q_5:

$$q_5 = 2^{\frac{5}{12}} = (\sqrt[12]{2})^5 = 1.3348.. \approx 1.\bar{3} = \frac{4}{3}$$

Definition 1.4.6 (Minor Third)**.** *The minor third of τ is the another interval we won't define as a scale adjusted harmonic. Rather, the minor third, denoted t is defined between the fifth harmonic $\mathbf{h}_5 = \square_2 T$ and the sixth harmonic $\mathbf{h}_6 = \square_2 \varphi$. So,*

$$t = \frac{\mathbf{h}_6}{\mathbf{h}_5}$$

*That being said, it is perhaps a more insightful notion to see that the minor third naturally occurs in the context of the **major** triad. Namely, we can also define it as the interval between between the major third and the perfect fifth. So,*

$$t = \frac{\varphi}{T} = \frac{3/2}{5/4} = \frac{6}{5}$$

This highlights the special relation, which we will find to be the genesis of major and minor chords:

$$tT = Tt = \varphi$$

So, given τ, we denote the minor third $t(\tau)$, meaning:

$$t(\tau) = \frac{6}{5}\tau$$

In 12-TET, the minor third is approximated by 3 semitones, the interval q_3:

$$q_3 = 2^{\frac{3}{12}} = \sqrt[4]{12} = 1.1892.. \approx 1.2 = \frac{6}{5}$$

(**Checkpoint: Justonic Interval Approximations**)

So, let us recap what we have learnt:

Interval	k	q_k	Symbol	Justonic Approximant	Ratio
Major 2^{nd}	2	$2^{2/12} = \sqrt[6]{2}$	w	$\mathbb{H}_9$	$\frac{9}{8}$
Minor 3^{rd}	3	$2^{3/12} = \sqrt[4]{2}$	t	$\varphi(\mathbb{H}_5)^{-1}$	$\frac{6}{5}$
Major 3^{rd}	4	$2^{4/12} = \sqrt[3]{2}$	T	$\mathbb{H}_5$	$\frac{5}{4}$
Perfect 4^{th}	5	$2^{5/12} = (\sqrt[12]{2})^5$	ρ	$\square_1(\mathbb{H}_3)^{-1}$	$\frac{4}{3}$
Tritone	6	$2^{6/12} = \sqrt{2}$	⫛	-	-
Perfect 5^{th}	7	$2^{7/12} = (\sqrt[12]{2})^7$	φ	$\mathbb{H}_3$	$\frac{3}{2}$

Chapter 2

Chromatic-Accidental Arithmetic

2.1 Chapter Summary

(**Summary - Melodic-Chromatic Arithmetic: Naming the Notes**)

2.2 Pitch Labeling: Naming the Notes

We start from the very beginning. In music, we often hear letters being thrown around. When it comes to notes, there are seemingly only 7 letters.

Consider the musical alphabet:

$$\mathcal{L} = \{A, B, C, D, E, F, G\}$$

Definition 2.2.1 (Musical Alphabet)**.** *The musical alphabet is defined as a set of letters $\mathcal{L} = \{A, B, \ldots, F, G\}$, where the letter after* **G** *loops back to be* **A** *again, and so on.*

Note ($\mathcal{L} \cong \mathbb{Z}_7$)**.** *$\mathcal{L}$ is isomorphic (equivalent in structure) to the group of remainders after dividing any integer from $\mathbb{Z}$ by 7, the set which we call the field $\mathbb{Z}_7 = \{0, 1, \ldots, 5, 6\}$. In other words, it is a relabeling of $\mathbb{Z}_7$. In this group, we can more clearly see the previously stated fact about* ***looping*** *if we express we fix*

$A = 0, B = 1, \ldots, G = 6$ and observe $6 + 1 = 7 \cong 0$ corresponds to G being followed by A.

2.2.1 Letter Degrees & Seqeuences

In turn, we are ready to define letter-scale degrees.

Definition 2.2.2 (k-Degree Letter)**.** *The k^{th} degree letter from a given letter $L \in \mathcal{L}$ is denoted the value $\Lambda_k(L)$, which is defined as $L \oplus k$, In other words, it is the letter obtained from taking the final element of the sequence $\mathbb{L}_k[L]$.*

Definition 2.2.3 (k-Degree Letter Sequence)**.** *The k^{th} degree letter sequence from a given letter $L \in \mathcal{L}$ is denoted the set $\mathbb{L}_k[L]$,*

$$\mathbb{L}_k[L] = \{L, L \oplus 1, \ldots, L \oplus (k-1)\}$$

Or, if we let $\Lambda_j(L) = L \oplus j$, the equivalent sequence:

$$\mathbb{L}_k[L] = \{\Lambda_j(L) \mid j \in \mathbb{N}\}_{j=0}^{k-1}$$

*where the $\oplus$ represents taking a number of $(k-1)$ more steps in the musical alphabet, **with looping**. The emphasis is in this small part here, which gives us non-trivial relations. As such, we obtain k letters in total if we include our original starting letter (L), so we call it the k^{th} scale degree.*

Example (5^{th} from F)**.** *The fifth-degree letter from F is the terminus of the sequence:*

$$\mathbb{L}_5[F] = \{F, G, A, B, C\}$$

Or in other words, $\Lambda_5[F] = C$.

Example (4^{th} from C)**.** *The fourth-degree letter from C is the terminus of the sequence:*

$$\mathbb{L}_4[C] = \{C, D, E, F\}$$

Or in other words, $\Lambda_4[C] = F$.

Example (3^{rd} from C)**.** *The third-degree letter from C is the terminus of the sequence:*

$$\mathbb{L}_3[C] = \{C, D, E\}$$

Or in other words, $\Lambda_3[C] = E$.

Example (6^{th} from E). *The sixth-degree letter from* E *is the terminus of the sequence:*

$$\mathbb{L}_6[E] = \{E, F, G, A, B, C\}$$

Or in other words, $\Lambda_6[E] = C$.

Example (7^{th} from A). *The seventh-degree letter from* A *is the terminus of the sequence:*

$$\mathbb{L}_7[A] = \{A, B, C, D, E, F, G\}$$

Or in other words, $\Lambda_7[A] = G$.

Example (2^{nd} from G). *The second-degree letter from* G *is the terminus of the sequence:*

$$\mathbb{L}_2[G] = \{G, A\}$$

Or in other words, $\Lambda_2[G] = A$.

Note that the patterns that are emerging might be surprising. They were in fact chosen intentionally, to show the following theorem.

Theorem 2.2.1 (Rule of 9). *The rule of nine tells us the following:*

$$\Lambda_k[L] = \Lambda_{9-k}[L'] \iff \Lambda_{9-k}[L'] = \Lambda_k[L]$$

To put it more simply, we pair fifths & fourths, sixths & thirds, and sevenths & seconds. This could be attributed to the octave-partitioning scale axiom. Since the union of these letter templates gives us diatonic scales.

Example (F-C-F). *The **fifth**-degree letter of* F *is the letter* C*. The **fourth**-degree letter of* C *is* F*. Indeed,* $5 + 4 = 9$*. Together, their union forms this particular diatonic letter scale partition:*

$$\{F, G, A, B, C\} \cup \{C, D, E, F\}$$

Example (C-E-C). *The **third**-degree letter of* C *is the letter* E*. The **sixth**-degree letter of* E *is* C*. Indeed,* $3 + 6 = 9$*. Together, their union forms this particular diatonic letter scale partition:*

$$\{C, D, E\} \cup \{E, F, G, A, B, C\}$$

Example (A-G-A)**.** *The* ***seventh****-degree letter of A is the letter G. The* ***second****-degree letter of G is A. Indeed,* $2 + 7 = 9$*. Together, their union forms this particular diatonic letter scale partition:*

$$\{A, B, C, D, E, F, G\} \cup \{G, A\}$$

L	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	$L_{(1)}$
A	B	C	D	E	F	G	A
B	C	D	E	F	G	A	B
C	D	E	F	G	A	B	C
D	E	F	G	A	B	C	D
E	F	G	A	B	C	D	E
F	G	A	B	C	D	E	F
G	A	B	C	D	E	F	G

(**Scales, Revisited**)

Recall that we defined a scale as a set of intervals that lie between a frequency and one octave above it. This takes the form of a (Do, ..., Do) set of sonorities in Solfege. This definition of a scale is quite handy, and is in fact intertwined with the reasoning behind choosing 7 letters in $\mathcal{L}$. In fact, it is by construction that we do since we want each letter once to occur in a scale until we reach the same letter again (representing the same pitch an octave above). We want our scales to end at the same note, an octave above. As such, this naturally leads to needing 8 notes in total to go back to the same letter.

Note (Etymology of Octave)**.** *Alas, we have obtained the etymology of the word octave! That is, the word octave has the root word* $8-$ *(oct-). This is connected since a 8-letter note sequence is a transposition of the musical alphabet – or in other words, a diatonic scale partitioning an octave!*

That being said, we now define a generic heptatonic scale template, which is merely a transposition of $\mathcal{L}$ starting on any letter.

Definition 2.2.4 (Heptatonic Scale Template)**.** *Suppose we have* $L \in \mathcal{L}$*. The hepatonic scale template is the letter sequence:*

$$\mathcal{S}_7[L] = \mathbb{L}_7[L] := \mathcal{L}_{(L)}$$

We could also denote $\mathbb{L}_7[L] = \mathcal{L}_{(L)}$*, which is the musical alphabet, beginning at L. As such, the scale goes up until it reaches the same pitch an octave above, where the next iteration of the sequence repeats (also an octave above).*

Example (F Heptatonic Scale)**.** *The F heptatonic scale template is given by:*

$$\mathcal{S}_7[F] = (F, G, A, B, C, D, E)$$

To notate a scale repeating an octave above, we use an index in parenthesis to indicate the number of octaves transposed above. So, for instance, the scale would repeat:

$$\mathcal{S}_7[F] = (F, G, A, B, C, D, E, F_{(1)}, G_{(1)}, \dots)$$

Note that this gives us the octave relationship between our notes. For any $L \in \mathcal{L}$,

$$\frac{\mathrm{Freq}(L_{(k)})}{\mathrm{Freq}(L)} = 2^k = \square_k$$

2.2.2 Accidentals

Definition 2.2.5 (Accidentals)**.** *An accidental* $\alpha \in \mathcal{A}$*, where* $\mathcal{A} = \{\flat, \natural, \sharp\}$ *is a symbol that introduces a notion of distance, or arithmetic to the space of the musical alphabet* $\mathfrak{L}$*. Namely, it does so in accordance to the semitonal distance function* d_1*, defined below.*

(**Pitch Labels**)

Finally, we are able to formally construct notes as we know them. Recall that we defined two important objects coming from two sets:

(1) Letters, from the **Musical Alphabet**;

$$L \in \mathcal{L} = \{A, B, \dots, F, G\}$$

(2) Accidentals, from the **Single-Accidental Alphabet**;

$$\alpha \in \mathcal{A} = \{\flat, \natural, \sharp\}$$

Combining these two constructs together naturally motivates the definition of a note-letter. Let us denote the space of pairs of letters and accidentals $\mathcal{L}^* = \mathcal{L} \times \mathcal{A}$

Definition 2.2.6 (Note-Letter)**.** *A pitch label, note-letter, or more formally, an accidentalized alphabet letter, is defined by pairing an alphabet letter* $L \in \mathcal{L}$ *and an accidental* $\alpha \in \mathcal{A}$*. Then, an accidentalized letter-note pair is simply the pairing* π*:*

$$\pi = (L, \alpha) = L^\alpha \text{ where } (L, \alpha) \in \mathcal{L}^*$$

Note (Accidental Neutralization)**.** *Note that flats and sharps neutralize each other. In other words, any note-letter* L *satisfies* $(L^\flat)^\sharp = L^\natural = (L^\sharp)^\flat$

2.3 White-Key Note-Letter Distances

Before we begin, we formalize a notion mentioned above in our discussion of heptaonic scales.

Distance Rule 1 (Octave Distance). *The octave distance is defined as a binary operation $d_{\square} : \mathbb{Z} \to \mathbb{Z}$. In other words, it tells us the distance between a **fixed** alphabet letter $L \in \mathcal{L}$, and an $k \in \mathbb{Z}$ octave-transposed version of that same letter, $L_{(k)}$. Namely, we have:*

$$d_{\square}(L, L_{(k)}) = 12k$$

In other words, it's the linear function $f(k) = 12k$, where $k \in \mathbb{Z}$ is the number of octaves transposed. It basically just tells us, go down or up by 12 or -12 semitones for each octave. This rule is not super important, but we define it for completion.

Distance Rule 2 (Semitonal-Accidental Norm). *The semitonal distance is defined as a binary operation $d_{\alpha} : \mathcal{A} \to \{-1, 0, 1\}$. In other words, it tells us the distance between a **fixed** alphabet letter $L \in \mathcal{L}$, and an accidentalized version of that same letter, $L^{\alpha} \in \mathcal{L}^*$. Fixing $\alpha \in \mathcal{A}$:*

$$d_{\alpha}(L, L^{\alpha}) = |\alpha|$$

where we have:

$$|\alpha| = \begin{cases} -1, & \alpha = \flat \\ 0, & \alpha = \natural \\ +1, & \alpha = \sharp \end{cases}$$

Or in other words:

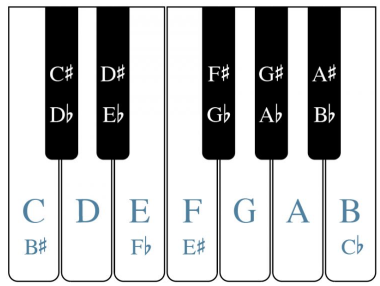
$$|\flat| = -1, |\natural| = 0, |\sharp| = +1$$

Example (C, $C^{\sharp}$). *The distance between the notes C and $C^{\sharp}$ is $|\sharp| = +1$. So, $d_{\alpha}(C, C^{\sharp}) = 1$.*

Example (D, $D^{\flat}$). *The distance between the notes D and $D^{\flat}$ is $|\flat| = -1$. So, $d_{\alpha}(D, D^{\flat}) = -1$.*

2.3.1 Accidentals & Color

Consider the diagram below:



Definition 2.3.1 (Black Flat-Supported Letter)**.** *A letter $L \in \mathcal{L}$ is called a black flat-supported letter if $Color(L^\flat) = \boldsymbol{Blk}$. That is,*

$$L \in \overline{\boldsymbol{Blk}}[\flat] \iff Color(L^\flat) = \boldsymbol{Blk}$$

The flat-supported keys are denoted $\overline{\boldsymbol{Blk}}[\flat]$ and are:

$$\overline{\boldsymbol{Blk}}[\flat] = \{A, B, D, E, G\}$$

And their complement is the white flats, $\overline{\boldsymbol{Wht}}[\flat]$:

$$\overline{\boldsymbol{Wht}}[\flat] = \mathcal{L} \setminus \overline{\boldsymbol{Blk}}[\flat] = \{C, F\}$$

Note (Flat Penatonics)**.** *Note that $\overline{\boldsymbol{Blk}}[\flat]$ could be more easily thought of as the notes that spell out $\boldsymbol{G}^\flat$ pentatonic major or $E^\flat$ pentatonic minor.*

Definition 2.3.2 (Black Sharp-Supported Letter)**.** *A letter $L \in \mathcal{L}$ is called a black sharp-supported letter if $Color(L^\sharp) = \boldsymbol{Blk}$.*

$$L \in \overline{\boldsymbol{Blk}}[\sharp] \iff Color(L^\sharp) = \boldsymbol{Blk}$$

The sharp-supported keys are denoted $\overline{\boldsymbol{Blk}}[\sharp]$ and are:

$$\overline{\boldsymbol{Blk}}[\sharp] = \{A, C, D, F, G\}$$

And their complement is the white sharps, $\overline{\boldsymbol{Wht}}[\sharp]$:

$$\overline{\boldsymbol{Wht}}[\sharp] = \mathcal{L} \setminus \overline{\boldsymbol{Blk}}[\sharp] = \{B, E\}$$

Note (Sharp Penatonics). *Note that* $\overline{\textbf{\textit{Blk}}}[\sharp$ *could be more easily thought of as the notes that spell out* $F^\sharp$ *pentatonic major or* $D^\sharp$ *penatonic minor.*

Definition 2.3.3 (Den Letter). *A letter* $L \in \overline{\textbf{\textit{Den}}}$ *is a den key if*

$$Color(L^\flat) = Color(L^\sharp) = \textbf{\textit{Blk}}$$

So, L is a den key when $L \in \overline{\textbf{\textit{Blk}}}[\flat]$ *and* $L \in \overline{\textbf{\textit{Blk}}}[\sharp]$, *meaning it is both black-sharp and black-flat supported. Or in other words,* $L \in \overline{\textbf{\textit{Blk}}}[\flat] \cap \overline{\textbf{\textit{Blk}}}[\sharp]$, *the intersection of the two sets. The den keys are denoted* $\overline{\textbf{\textit{Den}}}$ *and are:*

$$\overline{\textbf{\textit{Blk}}}_D = \{A, D, G\}$$

We call them den keys since the natural key versions look like dens in a piano, surrounded by two black keys.

We memorize the den keys with the mnemonic:

$$\overline{\textbf{Blk}}_D = \{A, D, G\} = \text{A Den “G”}$$

perhaps alluding to the fact that $G \in \overline{\textbf{Blk}}_D$.

That being said, we define the complement of the den keys to be the **white-neighbouring keys**

Definition 2.3.4 (White Neighbours). *A letter* $L \in \mathcal{L}$ *is a white neighbour if*

$$Color(L^\flat) \textbf{ or } Color(L^\sharp) = \textbf{\textit{Wht}}$$

So, L is a white neighbour ($Ll \in \overline{\textbf{\textit{WhtNeb}}}$) *when* $L \notin (\overline{\textbf{\textit{Blk}}}[\flat] \cap \overline{\textbf{\textit{Blk}}}[\sharp])$,

$$L \notin (\overline{\textbf{\textit{Blk}}}[\flat] \cap \overline{\textbf{\textit{Blk}}}[\sharp]) \iff L \notin (\overline{\textbf{\textit{Blk}}}[\flat]) \textbf{ or } L \notin (\overline{\textbf{\textit{Blk}}}[\sharp])$$

meaning it isn't both black-sharp and black-flat supported. The den keys are denoted $\overline{\textbf{\textit{WhtNeb}}} = \mathcal{L} \setminus \overline{\textbf{\textit{Den}}}$ *and are:*

$$\overline{\textbf{\textit{WhtNeb}}} = \{B, C, E, F\}$$

We call them den keys since the natural key versions look like dens in a piano, surrounded by two black keys.

That being said, we are finally able to compute distances between consecutive letters! Or in other words, start doing arithemetic exclusively on the **white notes** first.

2.3.2 Note Steps

Distance Rule 3 (Step Distance)**.** *The step distance is defined as another rule for the binary operation $d_2 : \mathcal{L} \to \{1, 2\}$. Consider a fixed letter $L \in \mathcal{L}$, and its right neighbour, or its second-degree letter $\Lambda_2 = L \oplus 1$. The distance between two consecutive, unaccidentalized alphabet letters can now computed using the above notions of black-accidental support.*

$$d_2(L) = d(L, \Lambda_2) = k \text{ where } 1 \leq k \leq 2$$

where we have:

$$d_2(L) = d(L, \Lambda_2) = \begin{cases} 1, & \Lambda_2 \in \overline{\boldsymbol{Wht}}[\flat] \\ 2, & \Lambda_2 \in \overline{\boldsymbol{Blk}}[\flat] \end{cases}$$

Equivalently,

$$d_2(L) = d(L, \Lambda_2) = \begin{cases} 1, & L \in \overline{\boldsymbol{Wht}}[\sharp] \\ 2, & L \in \overline{\boldsymbol{Blk}}_\sharp \end{cases}$$

So, notice the relationship:

$$L \in \overline{\mathbf{Wht}}[\sharp] \iff \Lambda_2 \in \overline{\mathbf{Wht}}[\flat]$$

$$L \in \overline{\mathbf{Wht}}[\flat] \iff \Lambda_2 \in \overline{\mathbf{Wht}}[\sharp]$$

Example (A, B)**.** *Since $A \in \overline{\boldsymbol{Blk}}[\sharp]$ or $B \in \overline{\boldsymbol{Blk}}[\flat]$, the distance $\delta_2(A) = \delta(A, B) = 2$.*

Example (E, F)**.** *Since $F \in \overline{\boldsymbol{Wht}}[\flat]$, the distance $\delta_2(E) = \delta(E, F) = 1$.*

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	B	2	w
B	C	1	h
C	D	2	w
D	E	2	w
E	F	1	h
F	G	2	w
G	A	2	w

Sorting them out,

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	B	2	w
C	D	2	w
D	E	2	w
F	G	2	w
G	A	2	w
B	C	1	h
E	F	1	h

Now, we are able to define enharmonics.

Definition 2.3.5 (Chromatic Enharmonic)**.** *The enharmonic of a note in between two letters a whole tone apart, accidentalized in the appopriate direction are equivalent and called proper enharmonics. They are:*

$$\{(A^{\sharp}, B^{\flat}), (C^{\sharp}, D^{\flat}), (D^{\sharp}, E^{\flat}), (F^{\sharp}, G^{\flat}), (G^{\sharp}, A^{\flat})\}$$

The white-key (improper) enharmonics are:

$$\{(B, C^{\flat}), (E, F^{\flat}), (B^{\sharp}, C), (E^{\sharp}, F)\}$$

Den Keys. Notice the role of den keys here!!!

In general, we can now define a general distance function δ for any degree letter.

Distance Rule 4 (n-Degree Distance)**.** *The n^{th}-degree distance is defined as another rule for the binary operation $\delta : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}_{12}$. Consider a fixed, unaccidentalized letter $L \in \mathcal{L}$, and its n^{th}-degree letter $\Lambda_n = L \oplus (n-1)$. We obtain its distance by iteratively finding the distances between the two steps along the way. Letting $L = \Lambda_0$,*

$$\delta_n(L) = \delta(L, \Lambda_n) = \sum_{j=0}^{n-1} d_2(\Lambda_j) = \sum_{j=0}^{n-1} d(\Lambda_j, \Lambda_{j+1})$$

2.4 Tertiary Harmony

2.4.1 Thirds/Skips and Dyads

(Dyads and Triads)

Finally, we are ready to define quality dyads - one of the most useful objects you will ever encounter. As we know, the words **major** and **minor** are ubiquitous in music theory. We will refer to them as tonal "qualifiers" where they add a quality to some tonal center. What is a note without its third?

Definition 2.4.1 (Quality)**.** *Quality is a binary indicator that tells us whether a chord is major or minor. This is only unambiguous for dyads and triads. We denote a quality* $\mathcal{Q} \in \tilde{\mathcal{Q}}$ *where* $\tilde{\mathcal{Q}} = \{\Delta, \nabla\}$.

By composing two steps, we obtain a skip.

Distance Rule 5 (Skip Distance)**.** *The skip distance is defined as another rule for the binary operation* $d_3 : \mathcal{L} \to \{3, 4\}$. *Consider a fixed, unaccidentalized letter* $L \in \mathcal{L}$*, and its third degree letter* $\Lambda_3 = L \oplus 2$. *We obtain its distance by iteratively finding the distances between the two steps along the way.*

$$d_3(L) = d(L, \Lambda_3) = k \text{ where } 3 \leq k \leq 4$$

where we have:

$$d_3(L) = d(L, \Lambda_3) = d_2(L, \Lambda_2) + d_2(\Lambda_2, \Lambda_3)$$

Recall that d_2 *has a range of* $\{1, 2\}$*, since no two consecutive letters lie in* $\overline{\textbf{\textit{Wht}}}[\flat]$ *(we cannot find 3 consecutive white keys in a piano, only a maximum of 2),* d_3 *has a range of* $\{3, 4\}$. *Recall* $\overline{\textbf{\textit{Wht}}}[\flat] = \{C, F\}$ *so regardless of what we choose as* L*, the smallest degree we can get where both letters are in* $\overline{\textbf{\textit{Wht}}}[\flat]$ *is* 4.

Next up, classifying dyads:

$$\mathcal{Q}_3(L) = \mathcal{Q}(L, \Lambda_3) = \begin{cases} \nabla, & d(L, \Lambda_3) = 3 \\ \Delta, & d(L, \Lambda_3) = 4 \end{cases}$$

Well, since we are working with a few number of steps, there are a small enough number of possiblities such that we could conceieveably count them. If we want to classify the ways to obtain skips from steps, we can only use the following arithemtic partitions of the numbers 3 and 4 (allowing only 1 and 2)

$$3 = 1 + 2 = 2 + 1$$

$$4 = 2 + 2$$

Corresponding to the interval patterns:

$$t = hw = wh$$

$$T = ww = w^2$$

L	Λ_3	$d_3(L, \Lambda_3)$	Symbol
A	C	3	∇
B	D	3	∇
C	E	4	Δ
D	F	3	∇
E	G	3	∇
F	A	4	Δ
G	B	4	Δ

Sorting them out,

L	Λ_3	$d_3(L, \Lambda_3)$	Symbol
A	C	3	∇
B	D	3	∇
D	F	3	∇
E	G	3	∇
C	E	4	Δ
F	A	4	Δ
G	B	4	Δ

Definition 2.4.2 (Quality Dyad)**.** *Consider a letter $L \in \mathcal{L}$. A quality dyad is the pair of a note-letter on L and its third $T = \Lambda_3(L)$. In other words:*

$$\pi = (L^{\alpha}, T^{\alpha_T})$$

In other words, it is an assignment of an incidental pair (α, α_T) onto a pair of notes, a root and its third. This induces the next rule for semitonal distance. Namely, if the quality of the quality dyad is major, then:

$$d(L^\alpha, T^{\alpha_T}) = 4 \implies \kappa_2\Delta[L^\alpha] = T^{\alpha_T}$$

Example (A, $C^\sharp$ Major Dyad)**.** *The A major dyad is given by the note pair $\pi = \kappa_2\Delta[A] = \Delta_2[A] = (A, C^\sharp)$.*

Definition 2.4.3 (Complimentary Dyad)**.** *Consider a letter $L \in \mathcal{L}$. If we have a quality dyad $\mathcal{Q}_2(L)$, then the complimentary dyad is the dyad $\mathcal{Q}^c(\Lambda_3)$. If $Q = \Delta$, its complementary quality is minor, which we denote $Q^c = \nabla$. Or vice versa, if $Q = \nabla$, then $Q^c = \Delta$. Recall that:*

$$|\mathcal{Q}^c| = 7 - |\mathcal{Q}|$$

Rule: A Triad is the sum of any dyad and its complement.

$$\nabla[A] \oplus \Delta[C] = \int_A^E C$$

Rule: A Quality Seventh Chord is a stack of a dyad and its complementary chain.

$$\nabla^7[A] = \nabla[A] \oplus \Delta[C] \oplus \nabla[E] = \int_A^E C \oplus \int_C^G E$$

Alternatively, using triad-bass pairs:

$$\nabla^7[A] = \nabla[A] \oplus \Delta_3[C] = A \setminus \Delta_3[C]$$

Definition 2.4.4 (Quality Triad)**.** *Consider a letter $L \in \mathcal{L}$. A quality triad is the triple of a note-letters on L including its third $T = \Lambda_3(L)$ and fifth $F = \Lambda_5(L)$. In other words:*

$$\pi = (L^\alpha, T^{\alpha_T}, F^{\alpha_F})$$

In other words, it is an assignment of an incidental triple $(\alpha, \alpha_T, \alpha_F)$ onto a pair of notes, a root and its third. However, this time the letter F^{α_F} is predefined to be the complementary quality dyad to the root-third dyad, creating a perfect fifth. In other words, the major triad is given by:

$$\Delta_3 = (L, T = \Delta_2(L), F = \nabla_2(T))$$

And the minor triad:

$$\nabla_3 = (L, T = \nabla_2(L), F = \Delta_2(T))$$

Alluding to the fact that:

$$3 + 4 = 4 + 3 = 7$$

In summary:

$$Q_3 = (L, Q_2(L), Q_2^c(T))$$

2.4.2 White-Key Interval Complementation

(Steps and their Parity Complements)

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	B	2	w
B	C	1	h
C	D	2	w
D	E	2	w
E	F	1	h
F	G	2	w
G	A	2	w

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	$B^\flat$	1	h
C	$D^\flat$	1	h
D	$E^\flat$	1	h
F	$G^\flat$	1	h
G	$A^\flat$	1	h
B	$C^\sharp$	2	w
E	$F^\sharp$	2	w

(Thirds and their Quality Complements)

We do the same for thirds:

L	Λ_3	$d_3(L,\Lambda_3)$	Symbol
A	C	3	∇
B	D	3	∇
C	E	4	Δ
D	F	3	∇
E	G	3	∇
F	A	4	Δ
G	B	4	Δ

L	Λ_3	$d_3(L,\Lambda_3)$	Symbol
A	$C^\sharp$	4	Δ
B	$D^\sharp$	4	Δ
D	$F^\sharp$	4	Δ
E	$G^\sharp$	4	Δ
C	$E^\flat$	3	∇
F	$A^\flat$	3	∇
G	$B^\flat$	3	∇

2.4.3 Two Skips make a Fifth/Triads

By composing two thirds/skips, we obtain a fifth or a triad.

Distance Rule 6 (Triadic Distance)**.** *The triadic or fifth distance is defined as another rule for the binary operation $d_5 : \mathcal{L} \times \mathcal{L} \to \{6,7\}$. Consider a fixed, unaccidentalized letter $L \in \mathcal{L}$, and its fifth degree letter $\Lambda_5 = L \oplus 4 = \Lambda_2 \oplus 2$. We obtain its distance by iteratively finding the distances between the two skips/thirds along the way. Note, we only have diminished and perfect fifths between any two white keys. Augmented chords are not naturally occuring, in this sense.*

$$d_5(L) = d(L,\Lambda_5) = k \ \textit{where}\ 6 \le k \le 7$$

where we have:

$$d(L,\Lambda_5) = \begin{cases} 6 = |\{\nabla,\nabla\}| \\ 7 = |\{\nabla,\Delta\}| = |\{\Delta,\nabla\}| \end{cases}$$

In general, we use the equality:

$$d_5(L) = d(L, \Lambda_5) = d_3(L, \Lambda_3) + d_3(\Lambda_3, \Lambda_5)$$

L	Λ_5	$d_5(L, \Lambda_5)$	Symbol
A	E	7	φ
B	F	6	φ°
C	G	7	φ
D	A	7	φ
E	B	7	φ
F	C	7	φ
G	D	7	φ

Sorting them out,

L	Λ_5	$d_5(L, \Lambda_5)$	Symbol
A	E	7	φ
C	G	7	φ
D	A	7	φ
E	B	7	φ
F	C	7	φ
G	D	7	φ
B	F	6	φ°

And the corresponding complementary fixes/adjustments:

L	Λ_5	$d_5(L, \Lambda_5)$	Symbol
A	$E^\flat$	6	φ°
C	$G^\flat$	6	φ°
D	$A^\flat$	6	φ°
E	$B^\flat$	6	φ°
F	$C^\flat$	6	φ°
G	$D^\flat$	6	φ°
B	$F^\sharp$	7	φ

2.5 Fifths and Furry Cats

2.5.1 The Story of the Furry Cats

Remember the story of the Furry Cats:

Furry **C**ats **G**et **D**ancey **A**round **E**very **B**ird

Now, what do Furry Cats have to do with music theory? Well, this strange but cute mnemonic gives us an insight into how our white keys are tuned! Consider the following recursive chain of letter fifths:

$$F - C - G - D - A - E - B$$

In other words, we have the sequence:

$$\mathcal{F} = \{\lambda_0 = F, \lambda_1 = C, \dots, \lambda_6 = B\}$$

Using our letter cycle definition:

$$\mathcal{F} = \mathcal{C}_{7,5} = \{\lambda_{j+1} = \Lambda_5(\lambda_j) : \lambda_1 = F\}_{j=1}^{6}$$

Using index notation for compositions of letter degrees:

$$\mathcal{F} = \{\Lambda_5^{(c)}(F)\}_{c=0}^{6}$$

where $\Lambda_k^{(2)}(L) = \Lambda_k(\Lambda_k(L))$.

This chain is special, which we will call the furry-cat chain. It is special because it is the **unique** and **largest possible** chain of fifths that we can generate before needing **accidentals**. In other words, letting $\varphi = q_7 \in \mathbb{I}_{\mathcal{E}}[12]$ be our perfect fifth generator in our semitone space $\mathbb{Z}_{12}$, we obtain the canonical white key stack of fifths, which we call F Lydian.

Theorem 2.5.1 (Furry Cat Theorem)**.** *The Furry Cat Theorem states that the largest non-accidentalized possible chain of fifths is the sequence of white keys denoted by the Furry Cat Sequence. In other words, the Furry Cat sequence* $\mathcal{F} = \{\lambda_i = (L_i, \natural) \mid L_0 = F\}$ *where* $d_{\mathcal{T}}(L_i, L_{i-1}) = q_7 = \varphi$ *or more simply, have a* ***consecutive semitone distance*** *of a Perfect Fifth:*

$$d(L_i, L_{i-1}) = 7$$

Note (White-Note & Natural Accidentals)**.** *It is overdue to comment that any letter of the musical alphabet with the natural accidental* $\alpha = \natural$ *is considered a white note in the piano template.*

2.5.2 Circle of Fifths

Recall that $d_5(B, F) = 6$. By the semitone distance function, we know we could fix this by taking $F \mapsto F^\sharp$ so that we may have $d_5(B, F^\sharp) = d_5(B, F) + d_1(F, F^\sharp) = 6 + 1 = 7 = \varphi$

However, now, we have $d_5(F^\sharp, C) = d_5(F, C) - d_1(F, F^\sharp) = 7 - 1 = 6$. So, to readjust, we do $C \mapsto C^\sharp$. This chain reaction motivates the tile of fifths.

Finally, we obtain the god-foresaken circle of fifths. By using the accidental bridge principle, we have the chain:

$$\mathcal{F}^\flat - \mathcal{F}^\natural - \mathcal{F}^\sharp$$

And it yields the tiled circle of fifths:

$$F^\flat - C^\flat - G^\flat - D^\flat - A^\flat - E^\flat - B^\flat$$

$$F^\natural - C^\natural - G^\natural - D^\natural - A^\natural - E^\natural - B^\natural$$

$$F^\sharp - C^\sharp - G^\sharp - D^\sharp - A^\sharp - E^\sharp - B^\sharp$$

Note that we only have twelve unique notes, so there are enharmonic spellings. In summary:

$$\mathcal{F} = \mathbb{S}_5^{(6)}(F) = \Phi_6[F]$$

The circle of fifths shall therefore be called:

$$\mathcal{F}^* = (\mathcal{F}^\flat \cup \mathcal{F} \cup \mathcal{F}^\sharp) = \Phi_{12}$$

The circle of fourths:

$$\mathcal{R}^* = (\mathcal{R}^\sharp \cup \mathcal{R} \cup \mathcal{R}^\flat)$$

2.6 Chromatic Pitch Neighbours

2.6.1 Accidental Transposition

This whole time, we were discussing arithemtic on $\mathcal{L}$. How do we extend our arithemtic to $\mathcal{L}^*$? The answer? Mapping note letters from $\mathcal{L} \times \mathcal{L}$ that are distance-equivalent in $\mathcal{L}^* \times \mathcal{L}^*$.

Theorem 2.6.1 (Accidental-Transpotional Preservation Principle)**.** *Suppose we have two unaccidentalized letters $L, L_* \in \mathcal{L}$ which are a distance $\tilde{d}(L, L_*) = k$ semitones apart. Then, if we fix an accidental $\alpha \in \mathcal{A}$, accidentalizing both letters by the same accidental α preserves the distance. In other words,*

$$\tilde{d}(L, L_*) = k \iff \tilde{d}(L^\alpha, L_*^\alpha) = k$$

Note that here we use the function $\tilde{d}$, which automatically assumes the correct form depending on what degree $j \in \{1, 2, \dots, 7\}$ we have for $L_* = \Lambda_j(L)$. In a way, this theorem is a form of a **sufficiency** result on the white keys or unaccidentalized letter space.

Definition 2.6.1 (k-Degree Letter Stack)**.** *The k^{th} degree letter stack sequence on a root $L \in \mathcal{L}$ is denoted $\mathbb{S}_k^{(N)}[L]$ and is given by recursively taking N iterations of Λ_k starting at L. So, letting $\Lambda^{(0)} = L$:*

$$\mathbb{S}_k[L] = \{\Lambda_k^{(j+1)} = \Lambda_k(\Lambda_k^{(j)})\}_{j \in \mathbb{N}}$$

Example (Thirds from F)**.** *We have:*

$$\mathbb{S}_3[F] = \{F, A, C, E, G, B, D, F, A, C, \dots\}$$

2.6.2 Pitch Distances in Any Key

Definition 2.6.2 (Chromatic Left-Neighbour)**.** *The chromatic left-neighbour chord is the chord obtained by shifting every pitch in it down by one semitone, or effectively **flattening** it. So, given a chord κ with pitches:*

$$\kappa = \{\pi_1, \pi_2, \dots, \pi_N\}_{N \in \mathbb{N}}$$

We compactly write this:

$$\kappa \odot \flat = \{\pi_i \odot \flat\}_{i=1}^N$$

More simply,

$$\kappa \odot \flat = \{(\pi_1 \odot \flat), (\pi_2 \odot \flat), \dots, (\pi_N \odot \flat)\}$$

We similarly define right-neighours, as one might expect.

Definition 2.6.3 (Chromatic Right-Neighbour)**.** *The chromatic right-neighbour chord is the chord obtained by shifting every pitch in it down by one semitone, or effectively* ***sharpening*** *it. So, given a chord κ with pitches:*

$$\kappa = \{\pi_1, \pi_2, \dots, \pi_N\}_{N \in \mathbb{N}}$$

We compactly write this:

$$\kappa \odot \sharp = \{\pi_i \odot \sharp\}_{i=1}^{N}$$

More simply,

$$\kappa \odot \sharp = \{(\pi_1 \odot \sharp), (\pi_2 \odot \sharp), \dots, (\pi_N \odot \sharp)\}$$

2.6.3 Labelled Pitch Interval Arithmetic

Now, we have recently discussed the *Accidental-Transpotional Preservation Principle*. This means that we know how to spell many new intervals in $\mathcal{L}^*$. Now, we formalize this notion with pitch arithmetic notation. Of course, the interval math is still the same. There is nothing major changing. The only thing is that we are giving a whole set of specialized notation to drill in the different nature of this arithmetic, which is specialized to deal with note labels and ease the process of doing math with them.

Definition 2.6.4 (Pitch Product)**.** *Consider a labelled pitch $\pi \in \mathcal{L}^*$, and an interval $\iota \in \mathcal{E}_{12}$ with a frequency scalar κ. The product pitch is denoted $\pi \odot \iota$:*

$$\Pi_\iota(\pi) = \pi \odot \iota = \pi' \mid \pi' \in \mathcal{L}^*$$

which satisfies:

$$Freq(\pi') = \kappa \cdot Freq(\pi)$$

(**Fifths**)

From our white keys, we know that we have the chain of fifths by the furry cat sequence. So for example,

$$\Pi_{\varphi}(\pi) = \varphi(\pi) = \pi_7$$

We can now say some meaningful things. Suppose $\pi_k = \pi \odot q_k$. Then, we have that $\pi_7 = \varphi(\pi)$, the perfect fifth.
Perfect Fifth. Now, suppose Letter$(\pi_0) \neq B$ is any letter $L \in \mathcal{L} \setminus \{B\}$. Then, due to the **tiling semitone bridge gap** occuring on B:

$$\text{Letter}(\pi_0) \neq B \iff \text{Acc}(\pi_7) = \text{Acc}(\pi_0)$$

How about the notation below?

$$\text{Letter}(\square) \neq B \iff \text{Acc}(\varphi) = \text{Acc}(\square)$$

Whereas if the initial letter is B,

$$\text{Letter}(\pi_0) = B \iff \text{Acc}(\pi_7) = \text{Acc}(\pi_0) \odot \sharp$$

How about the notation below?

$$\text{Letter}(\square) = B \iff \text{Acc}(\varphi) = \text{Acc}(\square) \odot \sharp$$

Note. Sufficiency of Furry Cat Theorem and the Chromatic Theorem (writing intervals by memorizing white-key tables and applying the Chromatic Algorithm) give the minimum, sufficient theoretical basis to write tonal seventh chords very easily.

Use chromatic algorithm (i.e. the music theory of borrowing, like in 3rd grade addition) to find the qualities of intervals. Alternatively, can use chromatic neighbours to memorize patterns at large. For example, $A - F$ is a minor sixth implies $A^{\flat} - F$ is a major one. Flattening the floor or raising the ceiling augment the interval by $+1$. Raising the floor or lowering the ceiling diminishes it by -1. The accidentals effect is dependent on whether we are putting it on the root or the (find standardized name, like addend, subtractand)

We know $A - F$ is a minor sixths since $F = \Lambda_6(A)$ and A is the white-key Aeolian scale.

In other words, B fifths tend to have mismatched accidentals. More colloquially, B letters have cursed shell spellings.

$$\varphi(B^{\flat}) = F^{\natural}, \quad \varphi(B^{\natural}) = F^{\sharp}, \quad \varphi(B^{\sharp}) = F^{\mathbf{x}}$$

Perfect Fourth. Analogously:

$$\text{Letter}(\pi_0) \neq F \iff \text{Acc}(\pi_5) = \text{Acc}(\pi_0)$$

$$\text{Letter}(\pi_0) = F \iff \text{Acc}(\pi_5) = \text{Acc}(\pi_0) \odot \flat$$

In the same vein of reasoning, F fourths tend to have mismatched accidentals. More colloquially, F letters have cursed fourth spellings.

$$\rho(F^{\natural}) = B^{\flat}, \quad \rho(F^{\sharp}) = B^{\natural}, \quad \rho(F^{\flat}) = B^{\flat\flat}$$

(Tritones)

So for tritones, we have:

$$\text{Letter}(\pi_0) \neq \boldsymbol{B} \iff \text{Acc}(\pi_0, \pi_6) = (\natural, \flat)$$

So above, we could have just used:

$$\mathbb{A} = (\natural, \flat)$$

(Steps: 1 + 1 = 2)

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	B	2	w
C	D	2	w
D	E	2	w
F	G	2	w
G	A	2	w
B	C	1	h
E	F	1	h

L	Λ_2	$d_2(L, \Lambda_2)$	Symbol
A	$B^\flat$	1	h
C	$D^\flat$	1	h
D	$E^\flat$	1	h
F	$G^\flat$	1	h
G	$A^\flat$	1	h
B	$C^\sharp$	2	w
E	$F^\sharp$	2	w

Half-Step. We notice a pattern by C major intrinsic bias.

$$\text{Letter}(\pi) \in \{B, E\} = \overline{\mathbf{Wht}}[\sharp] \iff \text{Acc}(\pi_1) = \text{Acc}(\pi_0) \odot \natural$$

So, if the letter is not black-sharp supported, it basically means we have two consecutive white keys. As such, they are both natural.

$$\text{Letter}(\pi) \in \{A, C, D, F, G\} = \overline{\mathbf{Blk}}_\sharp \iff \text{Acc}(\pi_1) = \text{Acc}(\pi_0) \odot \sharp$$

Whole-Step. Analogously,

$$\text{Letter}(\pi) \in \{B, E\} = \overline{\mathbf{Wht}}[\sharp] \iff \text{Acc}(\pi_2) = \text{Acc}(\pi_0) \odot \sharp$$

So, if the letter is not black-sharp supported, it basically means we have two consecutive white keys, as we've already mentioned. So, to go a whole step above, we need to go to the sharps of our white tone right-neighbour.

$$\text{Letter}(\pi) \in \{A, C, D, F, G\} = \overline{\mathbf{Blk}}[\sharp] \iff \text{Acc}(\pi_2) = \text{Acc}(\pi_0) \odot \natural$$

(**Thirds: 2 + 2 = 3**)

Suppose $\Pi_T(\pi_0) = \pi_4$ is the major third above a pitch $\pi \in \mathcal{L}^*$. Recall the table of white thirds:

L	Λ_3	$d_3(L, \Lambda_3)$	Symbol
A	C	3	∇
B	D	3	∇
D	F	3	∇
E	G	3	∇
C	E	4	Δ
F	A	4	Δ
G	B	4	Δ

And the corresponding complementary fixes/adjustments:

L	Λ_3	$d_3(L, \Lambda_3)$	Symbol
A	$C^\sharp$	4	Δ
B	$D^\sharp$	4	Δ
D	$F^\sharp$	4	Δ
E	$G^\sharp$	4	Δ
C	$E^\flat$	3	∇
F	$A^\flat$	3	∇
G	$B^\flat$	3	∇

Major Third. We notice a pattern by **C major bias**.

$$\text{Letter}(\pi) \in \{C, F, G\} \iff \text{Acc}(\pi_4) = \text{Acc}(\pi_0) \odot \natural$$

$$\text{Letter}(\pi) \in \{A, B, D, E\} \iff \text{Acc}(\pi_4) = \text{Acc}(\pi_0) \odot \sharp$$

Minor Third. Analogously,

$$\text{Letter}(\pi) \in \{C, F, G\} \iff \text{Acc}(\pi_3) = \text{Acc}(\pi_0) \odot \flat$$

$$\text{Letter}(\pi) \in \{A, B, D, E\} \iff \text{Acc}(\pi_3) = \text{Acc}(\pi_0) \odot \natural$$

(**Accidental Family**)

Minor Thirds.

Letter	Left-Neighbors	Original	Right-Neighbours
L	$\flat \odot \kappa$	$\natural \odot \kappa$	$\sharp \odot \kappa$
A	$(A^\flat, C^\flat)$	(A, C)	$(A^\sharp, C^\sharp)$
B	$(B^\flat, D^\flat)$	(B, D)	$(B^\sharp, D^\sharp)$
C	$(C^\flat, E^{\flat\flat})$	$(C, E^\flat)$	$(C^\sharp, E)$
D	$(D^\flat, F^\flat)$	(D, F)	$(D^\sharp, F^\sharp)$
E	$(E^\flat, G^\flat)$	(E, G)	$(E^\sharp, G^\sharp)$
F	$(F^\flat, A^{\flat\flat})$	$(F, A^\flat)$	$(F^\sharp, A)$
G	$(G^\flat, B^{\flat\flat})$	$(G, B^\flat)$	$(G^\sharp, B)$

Major Thirds.

Letter	Left-Neighbors	Original	Right-Neighbours
L	$\flat \odot \kappa$	$\natural \odot \kappa$	$\sharp \odot \kappa$
A	$(A^\flat, C)$	$(A, C^\sharp)$	$(A^\sharp, C^{\sharp\sharp})$
B	$(B^\flat, D)$	$(B, D^\sharp)$	$(B^\sharp, D^{\sharp\sharp})$
C	$(C^\flat, E^\flat)$	(C, E)	$(C^\sharp, E^\sharp)$
D	$(D^\flat, F)$	$(D, F^\sharp)$	$(D^\sharp, F^{\sharp\sharp})$
E	$(E^\flat, G)$	$(E, G^\sharp)$	$(E^\sharp, G^{\sharp\sharp})$
F	$(F^\flat, A^\flat)$	(F, A)	$(F^\sharp, A^\sharp)$
G	$(G^\flat, B^\flat)$	(G, B)	$(G^\sharp, B^\sharp)$

Chapter 3

Meledo-Harmonic Arithmetic

3.1 Chapter Summary

(**Tonal Group Theory/Harmonic-Tonal Arithmetic: Organizing Tonal Centres and Keys**)

3.2 Tonality

3.2.1 Tonal Scales

We define the tonal scales, in the sense that they embody a tonal center and have a structure helpful for melody and harmony.

Before we do, we define a useful construct: the quality vector.

Definition 3.2.1 (Quality Vector)**.** *A quality vector of length N is a vector $v^{(N)} \in Q^n$ consisting of a sequence of n qualities. In other words:*

$$v^{(N)} = (Q_i)_{i=1}^{N} \mid Q_i \in \{\Delta, \nabla\}$$

Now, we define the core tonal intervals.

Definition 3.2.2 (Core Tonal Intervals)**.** *First, consider the set of core tonal intervals:*

$$\mathbb{I}_T = \{\square, w, \rho, \varphi\}$$

Namely, these provide a melodic stability using w and harmonic stability using ρ and φ.

Now, we define a tonal scale.

Definition 3.2.3 (Tonal Scale)**.** *Let $\pi \in \mathcal{L}^*$ be a pitch. The tonal scale $\overline{Diat}(Q_3, Q_6, Q_7)$ built on π is a function of a triple quality vector $v^{(3)}$ and contains the intervals:*

$$\mathbb{I} = \mathbb{I}_T \cup \{Q_3, Q_6, Q_7\}$$

So, the tonal scale tuned to π is the set of frequencies:

$$\overline{Diat}_\pi = \{\pi\iota \mid \iota \in \mathbb{I}\}$$

In fact, we have duality between the Ionian Major and Aeolian Minor modes in that:

$$\overline{\text{Diat}}_{Ion}(L) \implies \text{Deg}(\{\Lambda_3(L), \Lambda_6(L), \Lambda_7(L)\}) = (\natural, \natural, \natural)$$

$$\overline{\text{Diat}}_{Aeo}(L) \implies \text{Deg}(\{\Lambda_3(L), \Lambda_6(L), \Lambda_7(L)\}) = (\flat, \flat, \flat)$$

Which yields the canonical white-key/black-key relations:

$$\overline{\text{Diat}}_{Aeo}(A) = \{A, B, C, D, E, F, G\}$$

$$\overline{\text{Diat}}_{Ion}(A) = \{A, B, C^\sharp, D, E, F^\sharp, G^\sharp\}$$

Conversly,

$$\overline{\text{Diat}}_{Ion}(C) = \{C, D, E, F, G, A, B\}$$

$$\overline{\text{Diat}}_{Aeo}(C) = \{C, D, E^\flat, F, G, A^\flat, B^\flat\}$$

3.2.2 Major & Minor

(Aeolian Minor)

$$\{w, h, w, w, w, h, w, w\}$$

$$\{2, 1, 2, 2, 2, 1, 2, 2\}$$

$$\{2, 3, 5, 7, 8, 10, 0\}$$

$$\{w, t, \rho, \varphi, s, \upsilon, \square\}$$

$$\{w, \rho, S\} \overset{S}{\longleftrightarrow} \{S, \square, T\} \overset{T}{\longleftrightarrow} \{T, \varphi, V\}$$

$$\{2, 4, 6\} \overset{S}{\longleftrightarrow} \{6, 1, 3\} \overset{T}{\longleftrightarrow} \{3, 5, 7\}$$

(Ionian Major)

$$\{w, w, h, w, w, w, h\}$$

$$\{2, 2, 1, 2, 2, 2, 1\}$$

$$\{2, 4, 5, 7, 9, 11, 0\}$$

$$\{w, T, \rho, \varphi, S, V, \square\}$$

$$\{\rho, S, \square\} \overset{\square}{\longleftrightarrow} \{\square, T, \varphi\} \overset{\varphi}{\longleftrightarrow} \{\varphi, V, w\}$$

$$\{4, 6, 1\} \overset{\square}{\longleftrightarrow} \{1, 3, 5\} \overset{\varphi}{\longleftrightarrow} \{5, 7, 2\}$$

We begin by studying...

3.3 $\Phi_{\mathcal{S}}$

Before we begin, we will define shells.

Definition 3.3.1 (φ-Shells)**.** *A φ-shell on a ANLP $\aleph$ is simply the note and its fifth together. That is, fixing $\aleph \in \mathbb{A} \times \mathcal{A}$, it has a shell $\int_{[\aleph]}$*

$$\int_{[\aleph]} = \{\aleph, \varphi(\aleph)\}$$

Definition 3.3.2 (w-Shells)**.** *A w-shell on a ANLP $\aleph$ is simply the note, its fifth, and its whole tone together. That is, fixing $\aleph \in \mathbb{A} \times \mathcal{A}$, it has a w shell $\int_{[\aleph]}^{w}$*

$$\int_{[\aleph]}^{w} = \{\aleph, \varphi(\aleph), \varphi^2(\aleph)\} = \{\aleph, \varphi(\aleph), w(\aleph)\}$$

Since $w = \varphi^2$, we call this a double-shell as well.

Note (w-shells)**.** *These shells look like suspended 2 chords or the primary triad basis (three consecutive fifths).*

Definition 3.3.3 (Φ_S)**.** *A Φ_S is defined to be a stack of a fixed number of perfect fifths. As such, it has a single parameter k, the number of fifths. From which, we define a Φ_k to be the set:*

$$\Phi_k = \{\varphi^n \mid n \in (0, 1, 2, \dots, k)\}$$

We assume that a Φ_S is about $\square$, unless otherwise noted.

That being said, in practice, we will only go up to Φ_6. Each level must be mastered prior to going to the next one. The problem with music theory is that you are expected to do arithmetic in Φ_6. Some people can manage to do it, but everyone implicitly goes through the stages. It's just that the stages were never pointed out, or perhaps understood. However, since the nature of cyclical scales involve intersection, each scale recursively uses the prior one in many multiciplitous ways.

3.4 Diatonic Scales

Definition 3.4.1 (Diatonic μ-Scale)**.** *A diatonic scale is a heptatonic (7-note) scale in which every letter in the musical alphabet is assigned an incidental. In particular, it a seven note scale which includes pitches that are a 7-chain of perfect fifths. To indicate* ***where in the chain*** *the* ***root*** *is, we introduce a modal phase $\mu \in \mathbb{N}$. Since it is a finite chain of seven fifths, the possibilities are $0 \leq \mu \leq 6$.*

Given a modal phase $\mu \in [0, 6]_{\mathbb{N}}$, the diatonic-$\mu$ scale is given by the $\overline{Diat}(\pi) = \Phi_6(\Pi_{\varphi^\mu})$ where:

$$\overline{Diat}_\pi(\mu) = \{\varphi^{k-\mu}(\Pi) \mid k \in [0, 6]_{\mathbb{N}}\}$$

so that:

$$\varphi^0 = \pi\varphi^\mu$$

In summary, the μ phase tells us where in our $\bigcirc_\varphi$ our root $\pi = \varphi^\mu$ lies. If $\mu = 0$, then it is the beginning of the chain itself. With this definition, the F Lydian Scale is the $\mu = 0$ mode for the white keys. Meaning it is the white key scale with which ALL the notes are expressible as a number of fifths stacked above the tonic F. Remember, this is due to the Furry Cat theorem. As such, what is C major? C major is the classical white key scale, so where does it fall?

In fact, C major is the $\mu = 1$ diatonic $\mathcal{F}$-scale, since it has **exactly one** note that is not expressible as a stack of fifths, the perfect fourth F! So, the C Major Ionian scale is given by:

$$\overline{\text{Diat}}_C(1) = \{\varphi^{k-1}(C) \mid k \in [0, 6]_{\mathbb{N}}\}$$

$$\overline{\text{Diat}}_C(1) = \{\varphi^{-1}(C), \varphi^0 C, \varphi(C), \varphi^2(C), \dots, \varphi^5(C)\}$$

$$\overline{\text{Diat}}_C(1) = \{\rho(C), \Box_0 C, \varphi(C), w(C), \dots, V(C)\}$$

$$\overline{\text{Diat}}_C(1) = \{F, C, G, D, \dots, B\}$$

$$\overline{\text{Dorian}}_\pi = \overline{\text{Diat}}_\pi(-3) = [\pi] \odot_\Pi \{v, t, \rho, \Box_0, \varphi, w, S\}$$

$$\overline{\text{Dorian}}_\pi = \overline{\text{Diat}}_\pi(-3) = [\pi] \odot_\Pi \{\rho^3, \rho^2, \rho, \Box_0, \varphi, \varphi^2, \varphi^3\}$$

$$\overline{\text{Dorian}}_\pi = \overline{\text{Diat}}_\pi(-3) = [\pi] \odot_\Pi \{\mathbb{V}_{☾}, \mathbb{T}_{☾}, \rho, \Box_0, \varphi, w, \mathbb{S}_{☼}\}$$

ρ^3	ρ^2	ρ	$\Box_0$	φ	φ^2	φ^3
v	t	ρ	$\Box_0$	φ	w	S
$\mathbb{V}_{☾}$	$\mathbb{T}_{☾}$	ρ	$\Box_0$	φ	w	$\mathbb{S}_{☼}$

For example, consider D Dorian:

$$\overline{\text{Dorian}}_D = \overline{\text{Diat}}_F(3) = \overline{\text{Diat}}_C(2) = \overline{\text{Diat}}_G(1) = \overline{\text{Diat}}_D(0)$$

$$\overline{\text{Dorian}}_D = \overline{\text{Diat}}_F(3) = \overline{\text{Lydian}}_F$$

$$\overline{\text{Dorian}}_D = \overline{\text{Diat}}_C(2) = \overline{\text{Ionian}}_C$$

$$\overline{\text{Diat}}_D(-3) = [D] \odot_\Pi \{v, t, \rho, \Box_0, \varphi, w, S\}$$

$\square_0$	φ^1	φ^2	φ^3	φ^4	φ^5	φ^6

$\square_0$	$\boldsymbol{\varphi}$	w	$\boldsymbol{S}$	$\boldsymbol{T}$	$\boldsymbol{V}$	$\pitchfork$

$\square_0$	☼	☼^{2}	$\mathbb{S}_{☼}$	$\mathbb{T}_{☼}$	$\mathbb{V}_{☼}$	☾$^+$

(**Building Blocks:** Φ_0 **to** Φ_3)

$$\Phi_0 = \square$$
$$\Phi_1 = \int_{[\hat{1}]}$$
$$\Phi_2 = sus2$$
$$\Phi_3 = sus2add6 = (\int_{[\hat{1}]}, \int_{[\hat{2}]})$$

3.4.1 Tonality and Quality Tetrads

Consider the Major Tetrad on A:

$$\Delta[k_4](A) = \Delta[A], \nabla[C^\sharp], \Delta[E] = \{A, C^\sharp, E, G^\sharp\}$$

We could summarize this using shells:

$$\Delta[k_4](A) = \int_A^\varphi + \int_{C^\sharp}^\varphi = \{A, C^\sharp, E, G^\sharp\}$$

In other words:

$$\Delta[k_4](A) = \int_A^\varphi + \int_{\Delta[\Lambda_3(A)]}^\varphi = \{A, C^\sharp, E, G^\sharp\}$$

More generally:

$$\Delta[k_4](\aleph) = \int_\aleph^\varphi \cup \int_{\Delta[\Lambda_3(\aleph)]}^\varphi$$

Likewise for minor tetrads:

$$\nabla[k_4](\aleph) = \int_\aleph^\varphi \cup \int_{\nabla[\Lambda_3(\aleph)]}^\varphi$$

In summary:

$$\mathcal{Q}[k_4](\aleph) = \int_\aleph^\varphi \cup \int_{\mathcal{Q}[\Lambda_3(\aleph)]}^\varphi$$

Using dyad notation:

$$\mathcal{Q}_4(\aleph) = \int_\aleph^\varphi \cup \int_{\mathcal{Q}_2(\aleph)}^\varphi$$

3.4.2 Major-Minor Duality

In fact, we have duality between the Ionian Major and Aeolian Minor modes in that:

$$\overline{\text{Diat}}_{Ion}(L) \implies \text{Deg}(\{\Lambda_3(L), \Lambda_6(L), \Lambda_7(L)\}) = \natural$$
$$\overline{\text{Diat}}_{Aeo}(L) \implies \text{Deg}(\{\Lambda_3(L), \Lambda_6(L), \Lambda_7(L)\}) = \flat$$

Which yields the canonical white-key/black-key relations:

$$\overline{\text{Diat}}_{Aeo}(A) = \{A, B, C, D, E, F, G\}$$

$$\overline{\text{Diat}}_{Ion}(A) = \{A, B, C^\sharp, D, E, F^\sharp, G^\sharp\}$$

Conversely,

$$\overline{\text{Diat}}_{Ion}(C) = \{C, D, E, F, G, A, B\}$$
$$\overline{\text{Diat}}_{Aeo}(C) = \{C, D, E^\flat, F, G, A^\flat, B^\flat\}$$

Finally, we obtain an important result. Notice that the note degrees, 3-6-7 are all a fifth apart. In fact, they form a whole-tone shell on 6. $(6 - 7 - 3) = (1 - 2 - 5) \ominus 2$. This is a subset of the circle of fourths itself!

3.5 The Color-Accidental Bridge (Piano Patterns)

Incidental vectors!

$$(\natural, \sharp, \natural) \Rightarrow^\flat (\flat, \natural, \flat)$$

Appendix A

Algorithm Appendix

In this appendix, we will enumerate the various algorithms used in this thesis for the purpose of transparency and reproducibility.

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Appendix B

Consonance Appendix

(**Formal Mathematics of Quantifying Consonance**)

Warning (Unnecessaryness)**.** *This math is slightly more advanced than that in the previous section. Quite frankly, this section is probably a waste of a normal person's day. That is, unless you are feeling the energy to learn some number theory one odd day, you fucking weirdo. It's not too bad, but really, let this thereby relieve you of any guilt of just skipping to the conclusion. Seriously, don't waste any your day if you don't need to. The next chapter of this book is infinitely more useful than this. Man, I'm a terrible salesman.*

Note (Quick Number Theory)**.** *All you gotta know is one very seemingly trivial, but important theorem in math. It says, any natural number* $n = 0, 1, 2, 3, \dots$ *can be written as a product using only copies of a finite number of primes. Let's say it can factored using only the first k primes, then* $p_1, p_2, \dots, p_k$*. Of course, we are allowed to take more than one multiple of each prime, so we give each prime an exponent and match their labels/indices, and match each prime* p_j *with its power* π_j*. Note, here* π *is just another variable, like* x*, which we choose for the word "power", since "p" is "*π*" in Greek. So, we raise each prime* $p_j \in \mathbb{P}$ *to the power of* $\pi_j \in \mathbb{N}$*. Of course,* π_j *can be* 0*, which yields* $p_j^0 = 1$*, which means the* j^{th} *prime doesn't show up in this factorization. In any case, we write this:*

$$n = \prod_{j=1}^{k} p_j^{\pi_j} = p_1^{\pi_1} p_2^{\pi_2} \dots p_k^{\pi_k}$$

So for instance, the number 10:

$$10 = 2 \times 5 = 2^1 3^0 5^1 \implies (\pi_1, \pi_2, \pi_3) = (1, 0, 1)$$

There exists a complicated formula called Euler's consonance formula which takes an interval ratio $\iota \in [1,2]_{\mathbb{Q}}$ and assigns it a value, which orders the ratios by consonance.

Indeed, the formula for the most part does predict consonance in a pretty accurate, somewhat objective manner. This shows us that the principle of complexity being generally dissonant does have mathematical basis.

Namely,

$$\text{Eul} : \mathbb{Q}^+ \to \mathbb{N}$$

Define an interval $\iota_i \in \mathbb{Q}$. If $\text{Eul}(\iota_0) < \text{Eul}(\iota_1)$, Euler believes that this entails that ι_0 is more **consonant** than ι_1. If we factor $n \in \mathbb{N}$ as:

$$n = \prod_{k=1}^{r} p_k^{\pi_k} = p_1^{\pi_1} p_2^{\pi_2} \cdots p_r^{\pi_r} \mid p_k \in \mathbb{P}, \pi_k \in \mathbb{N}$$

Then the gradus function evaluates it as:

$$\text{Eul}(n) = 1 + \sum_{k=1}^{r} a_k(p_k - 1)$$

We extend the gradus function from $\mathbb{N}$ to $\mathbb{Q}$ by the relation:

$$\text{Eul}\left(\frac{x}{y}\right) = \text{Eul}(x \cdot y)$$

Warning (Euler's Name)**.** *I know I hate to use another European mathematicians name to name things. This dude has too many things named after him already. Like fucking Euler's number e.*

(**Results**)

Euler's gradus function ranks our major intervals as such:

$$\text{Eul}(\square_0) = 1$$
$$\text{Eul}(\square_1) = 2$$
$$\text{Eul}(\varphi) = 4$$
$$\text{Eul}(\rho) = 5$$
$$\text{Eul}(T) = 7$$

$$\mathrm{Eul}(S) = 7$$

$$\mathrm{Eul}(w) = 8$$

$$\mathrm{Eul}(V) = 10$$

Gr. II.	2:5.	*Gr. IIX.*	3:7.	3:64.	1:160.
1:2.	1:18.	1:14.	1:25.	1:256.	5:32.
Gr. III.	2:9.	2:7.	1:28.	*Gr. X.*	1:162.
1:3.	1:24.	1:30.	4:7.	1:42.	2:81.
1:4.	3:8.	2:15.	1:45.	3:14.	1:216.
Gr. IV.	1:32.	3:10.	5:9.	6:7.	8:27.
1:6.	*Gr. VII.*	5:6.	1:60.	1:50.	1:288.
2:3.	1:7.	1:40.	3:20.	2:25.	9:32.
1:8.	1:15.	5:8.	4:15.	1:56.	1:384.
Gr. V.	3:5.	1:54.	5:12.	7:8.	3:128.
1:5.	1:20.	2:27.	1:80.	1:90.	1:512.
1:9.	4:5.	1:72.	5:16.	2:45.	
1:12.	1:27.	8:9	1:81.	5:18.	
3:4.	1:36.	1:96.	1:108.	9:10.	
1:16.	4:9.	3:32.	4:27.	1:120.	
Gr. VI.	1:48.	1:128.	1:144.	3:40.	
1:10.	3:16.	*Gr. IX.*	9:16.	5:24.	
	1:64.	1:21.	1:192.	8:15.	

Note	**C**	**D**	**E**	**F**	**G**	**A**	**B**	**C**
SolFa	**Do**	**Re**	**Mi**	**Fa**	**So**	**La**	**Ti**	**Do**
Ratio	**1:1**	**9:8**	**5:4**	**4:3**	**3:2**	**5:3**	**15:8**	**2:1**
GV	**1**	**8**	**7**	**5**	**4**	**7**	**10**	**2**

Appendix C

Consonance Appendix

Appendix D

Mnemonics Appendix

Furry Cat Sequence.
Furry Cats Get Dancey Around Every Bird
Frederic Chopin Gave Death An Eerie/Enviable Ballad
Fruity Cereal Goes Down As Enjoyable Breakfast

F-C Shell.
Furry Cats
Frederic Chopin

C-G Shell.

G-D Shell.

D-A Shell.

A-E Shell.

E-B Shell.